Method of the unknown trial function: sharp lower bounds on Laplace eigenvalues

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(joint with Bartłomiej Siudeja)

University of Illinois

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Dirichlet eigenvalues

$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$

Faber–Krahn inequality

Among all plane domains,

$$\lambda_1 A \quad \text{is minimal for disk}$$

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Theorem (Laugesen–Siudeja 2010)

Among triangles, the higher eigenvalue sum

$$(\lambda_1 + \cdots + \lambda_n)D^2$$

is minimal for equilateral, for each $n = 1, 2, 3, \ldots$

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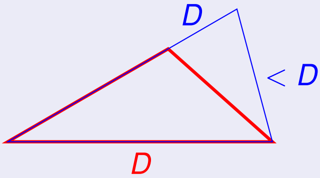

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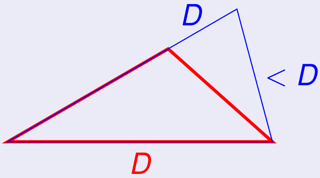
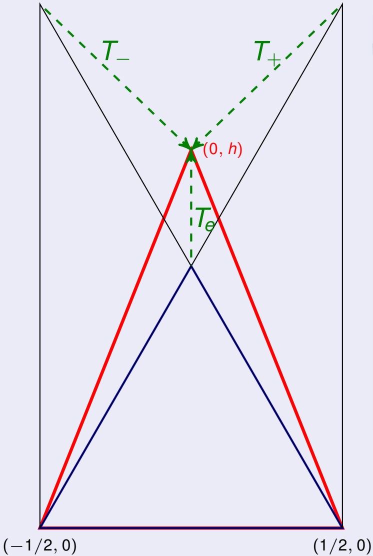


- Stretch arbitrary triangle to isosceles with same diameter.
- Eigenvalues decrease, by domain monotonicity.

Step 2 — Isosceles with aperture in $(\pi/6, \pi/3)$



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- Linearly map to isosceles Δ from equilateral and from 30-60-90° right triangles

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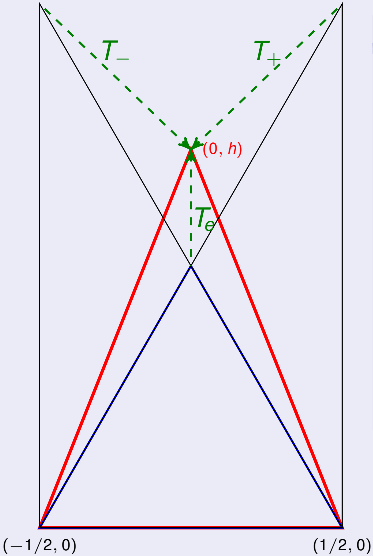

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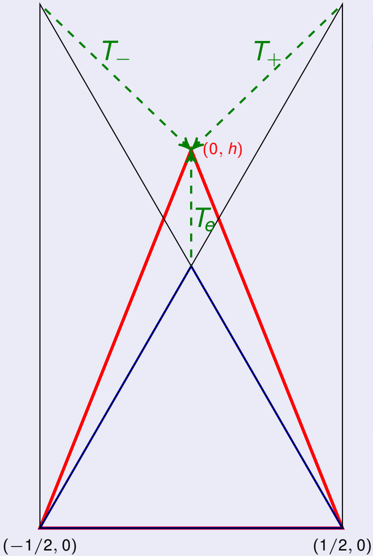

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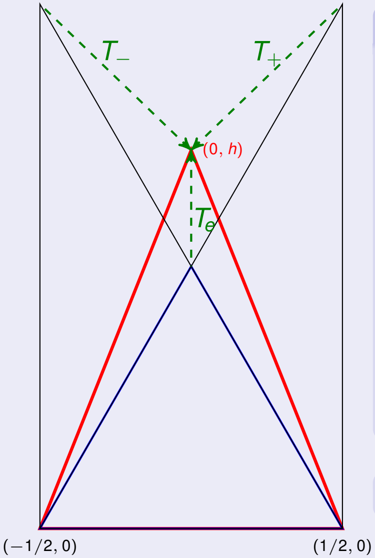

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  Also, on right side we really want $R[u_1] + \cdots + R[u_n]$.

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$$X = \sum_{j=1}^{n} \int_{\Delta} u_{j,x}^2 \, dA \qquad Y = \sum_{j=1}^{n} \int_{\Delta} u_{j,y}^2 \, dA$$

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We **want**

$$\lambda_1 + \cdots + \lambda_n|_{\triangle} \le (X + Y) D^2$$

where $D^2 = h^2 + (\frac{1}{2})^2$.

Define ratio

$$\gamma_n \stackrel{\text{def}}{=} \frac{(\lambda_1 + \cdots + \lambda_n)|_{◺}}{(\lambda_1 + \cdots + \lambda_n)|_{\triangle}}$$

Then previous slide says

$$\lambda_1 + \cdots + \lambda_n|_{\triangle} \leq X + \frac{4h^2}{3} Y$$

$$\lambda_1 + \cdots + \lambda_n|_{\triangle} \leq \left(\frac{13}{12} X + \frac{4h^2}{12} Y \right) \Big/ \gamma_n$$

We need at least one of right sides to be $< (X + Y)(h^2 + (\frac{1}{2})^2)$.

Thus we need

$$\frac{3}{4} \geq \frac{Y}{X+Y} \qquad \text{or} \qquad \frac{Y}{X+Y} \geq \frac{13 - 3\gamma_n(1+4h^2)}{13-4h^2}$$

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Equivalently, prove

$$\gamma_n = \frac{\lambda_1 + \cdots + \lambda_n|_{\text{◺}}}{\lambda_1 + \cdots + \lambda_n|_{\triangle}} \geq \frac{11}{24}$$

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Make rigorous using counting function, explicit formulas.

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- applies to linear transformations of arbitrary domains, not just triangles
- could be used on nonlinear transformations too?
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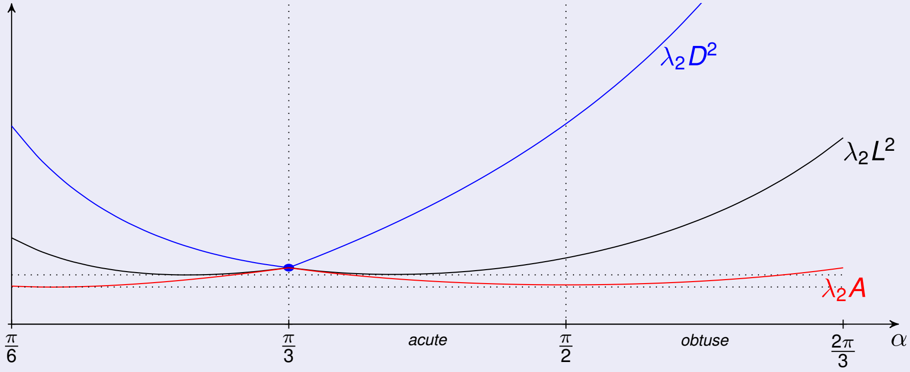

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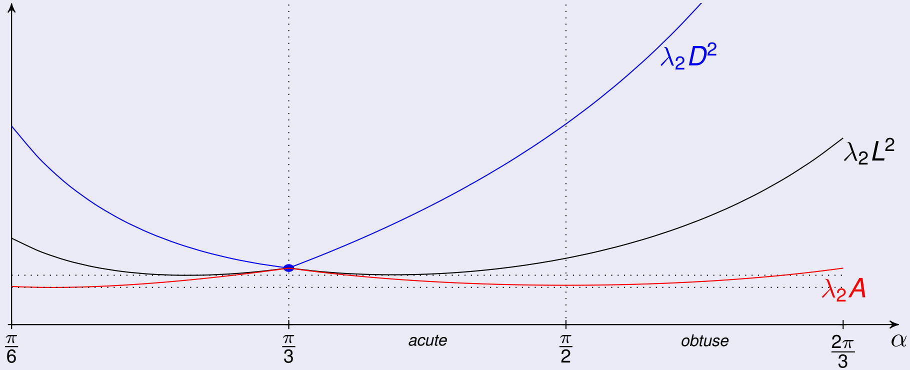

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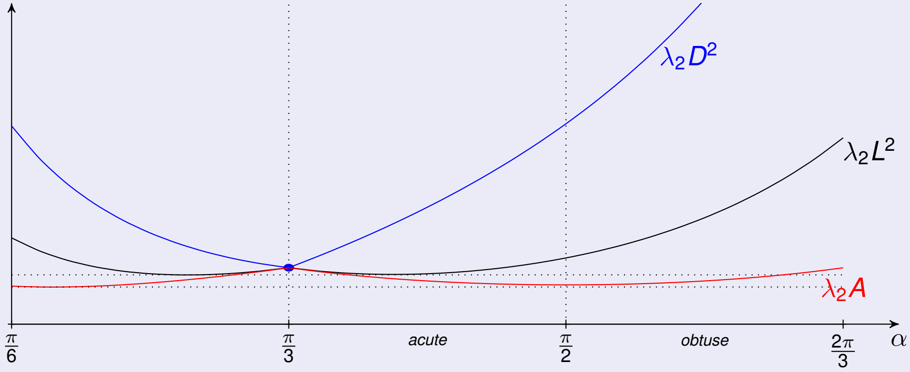

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(Consistent with general domains (Bucur, Henrot *et al*):
$\lambda_2 A$ and $\lambda_2 L^2$ minimal for stadium-like sets, not disk
$\lambda_2 D^2$ conjectured minimal for disk)

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and estimate $\lambda_1 + \lambda_2$ from below and λ_1 from above!

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Step 3. Subtract!

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- Spectral gap conjecture (van den Berg):
  Is $(\lambda_2 - \lambda_1)D^2$ minimal for degenerate rectangle?