On the lack of compactness in the 2D critical Sobolev embedding

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joint work with
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Outline

Introduction

The study of the lack of compactness in the Sobolev embedding has a long history. This question was investigated through several angles:

- P.-L. Lions[1985] : **Defect measures.**
- P. Gérard[1996-98]: **Microlocal defect measures and profile decomposition.**
- S. Jaffard[1999]: **Nonlinear wavelet approximation theory.**

Application

- **Qualitative study of nonlinear partial differential equations.**

- For $d \geq 3$, $0 < s < d/2$ and $p = 2d/(d-2s)$

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \quad \textbf{non-compact}$$

- **Translation and scaling invariance are the sole responsible for the defect of compactness** (P. Gérard).

$$(\tau_{y_n} u), \quad y_n \to \infty \ \text{ and } \ \delta_{h_n} u(\cdot) = h_n^{-\frac{d}{p}} u(\tfrac{\cdot}{h_n}), \quad h_n \to \infty \ \ or \ \ 0$$

- $\|\tau_{y_n} u\|_{L^p} = \|u\|_{L^p}, \ \ \|\delta_{h_n} u\|_{L^p} = \|u\|_{L^p}.$

The following profile decomposition was proved by P. Gérard

$$u_n(x) = u^0(x) + \sum_{j=1}^{\ell} \frac{1}{(h_n^{(j)})^{\frac{d}{p}}} \psi^{(j)} \left(\frac{x - x_n^{(j)}}{h_n^{(j)}} \right) + \mathrm{r}_n^{\ell}(x)$$

- u^0 is the weak limit.
- $(h_n^{(j)})$: scales, $(x_n^{(j)})$: cores, $(\psi^{(j)})$: profiles.
- **Orthogonality**: $j \neq k$

  - $h_n^{(j)}/h_n^{(k)} \to 0$ or $h_n^{(j)}/h_n^{(k)} \to \infty$
  - $h_n^{(j)} = h_n^{(k)}$ and $|x_n^{(j)} - x_n^{(k)}|/h_n^{(j)} \to \infty$.

- r_n^{ℓ} is small in L^p.

- **Stability**

$$\|u_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)}\|_{\dot{H}^s}^2 + \|\mathrm{r}_n^{(\ell)}\|_{\dot{H}^s}^2 + \circ(1), \quad n \to \infty.$$

- **L^p norm**

$$\|u_n\|_{L^p}^p \to \sum_{j\geq 1} \|\psi^{(j)}\|_{L^p}^p.$$

Applications

- **Qualitative study of the 3D critical NLW** (Bahouri-Gérard)

$$\partial_t^2 u - \Delta u + u^5 = 0$$

- **Blow-up analysis of the critical focusing NLW** (Kenig-Merle)

$$\partial_t^2 u - \Delta u - u^5 = 0$$

Critical 2D Sobolev embedding

Definition

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a convex increasing function such that*

$$\phi(0) = 0 = \lim_{s \to 0^+} \phi(s), \quad \lim_{s \to \infty} \phi(s) = \infty.$$

The Orlicz space L^ϕ *is defined via the Luxembourg norm*

$$\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

- **We may replace 1 by any positive constant.**
- $\phi(s) = s^p$, $1 \leq p < \infty \Longrightarrow L^\phi = L^p$.
- $\phi_\alpha(s) = \mathrm{e}^{\alpha s^2} - 1 \Longrightarrow L^{\phi_\alpha} = L^{\phi_1} = \mathcal{L}$.

Proposition

We have $H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2) \hookrightarrow \bigcap_{2\leq p<\infty} L^p(\mathbb{R}^2)$. *More precisely*

$$\|u\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1} \,. \tag{1}$$

- For $\alpha < 4\pi$ there exists C_α s.t.

$$\|\nabla u\|_{L^2} \leq 1 \Longrightarrow \left\| e^{\alpha|u|^2} - 1 \right\|_{L^1(\mathbb{R}^2)} \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)} \,.$$

- **The condition $\alpha < 4\pi$ is sharp.**
- $\alpha = 4\pi$ becomes admissible if we require $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$.

$$\sup_{\|u\|_{H^1}\leq 1} \int_{\mathbb{R}^2} (\mathrm{e}^{4\pi|u|^2} - 1)\, dx := \kappa < \infty$$

- The inequality (1) is insensitive to space translation but not invariant under scaling nor oscillations.
- The embedding of $H^1(\mathbb{R}^2)$ in $\mathcal{L}(\mathbb{R}^2)$ is sharp within the context of Orlicz spaces.
- $H^1(\mathbb{R}^2) \hookrightarrow BW(\mathbb{R}^2) \subsetneq \mathcal{L}(\mathbb{R}^2)$.
  Remark that the Brézis-Wainger space $BW(\mathbb{R}^2)$ is a rearrangement invariant Banach space but not an Orlicz space.
- $H^1(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.
  The spaces $\mathcal{L}$ and $BMO \cap L^2$ are not comparable.

A first analysis of the lack of compactness

The embedding $H^1 \hookrightarrow \mathcal{L}$ is non-compact at least for two reasons.

- **Lack of compactness at infinity**:
  $u_n(x) = \varphi(x + x_n), \ 0 \neq \varphi \in \mathcal{D}, \ |x_n| \to \infty$.
- **Concentration**: Lions's example

$$f_\alpha(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1, \\ -\frac{\log |x|}{\sqrt{2\alpha\pi}} & \text{if} \quad \mathrm{e}^{-\alpha} \leq |x| \leq 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-\alpha}. \end{cases}$$

Straightforward computations show that

- $\|f_\alpha\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{4\alpha}(1-\mathrm{e}^{-2\alpha}) - \frac{1}{2}\mathrm{e}^{-2\alpha}$.
- $\|\nabla f_\alpha\|_{L^2(\mathbb{R}^2)} = 1$.
- $f_\alpha \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ as $\alpha \to \infty$ or $\alpha \to 0$.
- $\|f_\alpha\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}}$ as $\alpha \to \infty$.
- $\|f_\alpha\|_{\mathcal{L}} \to 0$ as $\alpha \to 0$.

The difference between the behavior of f_α in Orlicz space when $\alpha \to 0$ or $\alpha \to \infty$ comes from the fact that the concentration effect is only displayed when $\alpha \to \infty$.

- $|\nabla f_\alpha|^2 \to \delta(x=0) \qquad (\alpha \to \infty)$.
- $\|f_\alpha\|_{\mathcal{L}} \sim \|f_\alpha\|_{L^2} \qquad (\alpha \to 0)$.
- In $H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ we have

$$\|\cdot\|_{\mathcal{L}} \sim \|\cdot\|_{L^2}$$

The following result of P.-L. Lions (in a slightly different form) characterizes the possible loss of compactness macroscopically.

Proposition

Let (u_n) *be a sequence in* $H^1(\mathbb{R}^2)$ *such that*

- $u_n \rightharpoonup 0$
- $\liminf_{n\to\infty} \|u_n\|_{\mathcal{L}} > 0$
- $\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n(x)|^2\, dx = 0$

Then, there exists $x_0 \in \mathbb{R}^2$ *and a constant* $c > 0$ *such that*

$$|\nabla u_n(x)|^2\, dx \rightharpoonup \mu \geq c\,\delta_{x_0} \quad (n \to \infty)$$

weakly in the sense of measures.

Main result

- We restrict ourselves to the radial case. The reason behind is the following well known L^∞ estimate

$$|u(x)| \leq \frac{C}{r^{\frac{1}{2}}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} .$$

- No defect of compactness far from the origin.
- The **fundamental** remark in our analysis is

$$f_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}}\, \mathbf{L}\left(\frac{-\log|x|}{\alpha}\right),$$

where

$$\mathbf{L}(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ t & \text{if} \quad 0 \leq t \leq 1, \\ 1 & \text{if} \quad t \geq 1. \end{cases}$$

- **The sequence $\alpha \to \infty$ is called the scale and the function L the profile.**
- $f_\alpha + f_{2\alpha} = \sqrt{\frac{\alpha}{2\pi}}\,\psi\left(\frac{s}{\alpha}\right)$ where

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ t + \frac{t}{\sqrt{2}} & \text{if} \quad 0 \leq t \leq 1, \\ 1 + \frac{t}{\sqrt{2}} & \text{if} \quad 1 \leq t \leq 2, \\ 1 + \sqrt{2} & \text{if} \quad t \geq 2. \end{cases}$$

- **The situation is completely different for $f_\alpha + f_{\alpha^2}$.**
- **We have $(\alpha)\perp(\alpha^2)$ and $(\alpha)\not\perp(2\alpha)$.**

- Our main goal is to establish that the characterization of the lack of compactness of the embedding

$$H^1_{rad} \hookrightarrow \mathcal{L}$$

can be reduced to the **Lions**'s example in terms of an asymptotic decomposition.

- In order to state our main result in a clear way, we need some definitions (Scales and Profiles).

Definition

A scale is a sequence $\underline{\alpha} := (\alpha_n)$ *of positive real numbers going to infinity. We shall say that two scales* $\underline{\alpha}$ *and* $\underline{\beta}$ *are orthogonal if*

$$\Big|\log\left(\beta_n/\alpha_n\right)\Big| \to \infty.$$

Definition

The set of profiles is

$$\mathcal{P} := \Big\{ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0 \Big\}.$$

- $\psi \in \mathcal{P} \implies \psi$ is continuous.
- If $\psi \in \mathcal{P}$ and $a \leq 0$ then $\psi_a(s) := \psi(s+a)$ belongs to $\mathcal{P}$.

Proposition

Let $\psi \in \mathcal{P}$ *a profile,* (α_n) *any scale and set*

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big).$$

Then

$$\frac{1}{\sqrt{4\pi}}\sup_{s>0}\frac{|\psi(s)|}{\sqrt{s}} \leq \liminf_{n\to\infty}\|g_n\|_{\mathcal{L}} \leq \limsup_{n\to\infty}\|g_n\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}}\|\psi'\|_{L^2}.$$

- If $\|\psi'\|_{L^2} = 1 = \sup\left(\frac{|\psi(t)|}{\sqrt{t}}\right)$ there exists $s_0 > 0$ such that $\psi(s) = \psi(s_0) = \sqrt{s_0}$ for any $s \geq s_0$.

Theorem

Let (u_n) *be a sequence in* $H^1_{rad}(\mathbb{R}^2)$ *such that*

$$u_n \rightharpoonup 0, \quad \limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0,$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{L^2(|x|>R)} = 0.$$

Then (up to subsequence extraction), for all $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x),$$

with

$$\limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \stackrel{\ell\to\infty}{\longrightarrow} 0.$$

- $(\alpha_n^{(j)})$ = scales, $\alpha_n^{(j)} \longrightarrow \infty$ as $n \longrightarrow \infty$.
- $(\alpha_n^{(j)}) \perp (\alpha_n^{(k)})$ for all $j \neq k$.
- $(\psi^{(j)})$ = profiles.
- **Stability**

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|(\psi^{(j)})'\|_{L^2}^2 + \|\nabla \mathrm{r}_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \longrightarrow \infty.$$

- **Orlicz norm**

$$\|u_n\|_{\mathcal{L}} \to \sup_{j\geq 1} \big(\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}} \big).$$

- **Diagonal subsequence extraction.**
- **Crucial fact:** Under the assumptions of the theorem we can extract a scale (α_n) and a profile ψ such that

$$\|\psi'\|_{L^2} \geq C\, A_0\,.$$

- **Study of the remainder term** r_n: **If** $\|\mathrm{r}_n\|_{\mathcal{L}} \to 0$ we stop the process; if not, r_n satisfies the same properties as u_n.
- **By contradiction arguments, we get the property of orthogonality between the two first scales.**
- **This process converges.**

- **Compactness at infinity** $\Longrightarrow \|u_n\|_{L^2} \longrightarrow 0.$
- **Radial setting**

$$\forall \;\; M \in \mathbb{R}, \quad \|v_n\|_{L^\infty(]-\infty, M[)} \longrightarrow 0 \;\; (n \to \infty), \quad v_n(s) = u_n(\mathrm{e}^{-s}).$$

- **As a consequence**

$$\forall \;\; \delta > 0, \quad \sup_{s \geq 0} \left(\left| \frac{v_n(s)}{A_0 - \delta} \right|^2 - s \right) \longrightarrow \infty \;\; (n \to \infty).$$

If not

$$\limsup_{n \to \infty} \|u_n\|_{\mathcal{L}} \leq A_0 - \delta.$$

- **Extraction of the first scale $\alpha_n^{(1)}$**

$$\frac{A_0}{2}\sqrt{\alpha_n^{(1)}} \leq |v_n(\alpha_n^{(1)})| \leq C\sqrt{\alpha_n^{(1)}} + \circ(1).$$

- **Extraction of the first profile $\psi^{(1)} \in \mathcal{P}$**

$$\psi_n(y) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\, v_n(\alpha_n^{(1)} y).$$

- $\psi_n' \rightharpoonup (\psi^{(1)})'$ in $L^2(\mathbb{R})$ with $\|(\psi^{(1)})'\|_{L^2} \geq \frac{\sqrt{2\pi}}{2} A_0.$
- $\|\psi^{(1)}(1)| = \left|\int_0^1 (\psi^{(1)})'(\tau)\, d\tau\right| \leq \|(\psi^{(1)})'\|_{L^2}.$

- $\mathrm{r}_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \left(\psi_n \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) - \psi^{(1)} \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) \right).$

$$\limsup_{n\to\infty} \|\nabla \mathrm{r}_n^{(1)}\|_{L^2}^2 \leq \limsup_{n\to\infty} \|\nabla u_n\|_{L^2}^2 - \|(\psi^{(1)})'\|_{L^2}^2 .$$

- **Let**

$$A_1 = \limsup_{n\to\infty} \|\mathrm{r}_n^{(1)}\|_{\mathcal{L}}.$$

If $A_1 = 0$ we are done. If not, we argue similarly to obtain a second scale $\alpha_n^{(2)}$ with $(\alpha_n^{(2)} \perp \alpha_n^{(1)})$.

- By iteration

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x)\,.$$

$$\limsup_{n\to\infty} \|\nabla\, \mathrm{r}_n^{(\ell)}\|_{L^2}^2 \le \limsup_{n\to\infty} \|\nabla\, u_n\|_{L^2}^2 - \sum_{j=1}^{\ell} \|(\psi^{(j)})'\|_{L^2}^2\,.$$

- We have $\|(\psi^{(j)})'\|_{L^2}^2 \ge C A_{j-1}$ for some absolute constant.
- It follows that

$$\limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{H^1}^2 \le \limsup_{n\to\infty} \|\nabla\, u_n\|_{L^2}^2 - C(A_0^2 + A_1^2 + \cdots + A_{\ell-1}^2)\,.$$

- Hence

$$A_\ell \to 0 \quad \text{as} \quad \ell \to \infty\,.$$

Qualitative study of nonlinear wave equation

Consider the following semi-linear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u + f(u) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x^2 \to \mathbb{R},$$

where

$$f(u) = u \, \left(\mathrm{e}^{4\pi u^2} - 1\right).$$

- Conservation of energy

$$\begin{aligned} E(u,t) &= \|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4\pi} \, \|\mathrm{e}^{4\pi u(t)^2} - 1\|_{L^1} \\ &= E(u,0) := E_0. \end{aligned}$$

- The notion of criticality here depends on the size of the initial energy E_0 with respect to 1.

- Nakamura-Ozawa: **Global well-posedness and scattering for sufficiently small data**.
- Atallah: **Local well-posedness for radially symmetric initial data** $(0, u_1)$.
- Ibrahim-Majdoub-Masmoudi & Ibrahim-Majdoub-Masmoudi-Nakanishi: **Global well-posedness and scattering in both subcritical and critical cases. Weak ill-posedness in the supercritical case**.
- **Very recently**, Struwe **has constructed global smooth solutions with radially symmetric data. Although the techniques are different, this result might be seen as an analogue of** Tao**'s result for the 3D energy supercritical wave equation**.

- We investigate the feature of solutions of the nonlinear Klein-Gordon equation taking into account the different regimes.
- Similar works of Gérard and Bahouri-Gérard.
- The approach that we adopt here is the one introduced by Gérard which consists to compare the evolution of oscillations and concentration effects displayed by sequences of solutions of the nonlinear Klein-Gordon equation and solutions of the linear Klein-Gordon equation.
- Roughly speaking, in the subcritical regime the nonlinear equation is linearizable. However, in the critical regime a nonlinear behavior appears.

Let $(\varphi_n, \psi_n) \in H^1 \times L^2$ **supported in a fixed compact and satisfying**

$$\varphi_n \rightharpoonup 0 \quad \text{in } H^1, \quad \psi_n \rightharpoonup 0 \quad \text{in } L^2 .$$

Definition

Let $T > 0$. *We shall say that the sequence* (u_n) *is linearizable on* $[0, T]$, *if*

$$\sup_{t \in [0,T]} E_c(u_n - v_n, t) \longrightarrow 0 \quad (n \to \infty)$$

where

$$E_c(w, t) = \int_{\mathbb{R}^2} \left[|\partial_t w|^2 + |\nabla_x w|^2 + |w|^2 \right] (t, x) \, dx .$$

Define

$$E^n = \|\psi_n\|_{L^2}^2 + \|\nabla\varphi_n\|_{L^2}^2 + \frac{1}{4\pi}\,\|\mathrm{e}^{4\pi\varphi_n^2} - 1\|_{L^1}\ .$$

Theorem

If $\limsup_{n\to\infty} E^n < 1$, *then* (u_n) *is linearizable.*
If $\limsup_{n\to\infty} E^n = 1$, *then* (u_n) *is linearizable provided that the sequence* (v_n) *satisfies*

$$L := \limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}.$$

We believe that the converse is true (work in progress).

We give in the sequel the ideas of the proof in the critical case.
Define $w_n = u_n - v_n$ and remark that

$$\partial_t^2 w_n - \Delta w_n + w_n = -f(u_n) \,.$$

We have to prove that $\|f(u_n)\|_{L^1([0,T];L^2(\mathbb{R}^2))} \to 0$.
The main tools to carry out the proof are:

- **Energy and Strichartz estimates.**
- **Convergence in measure.**
- **Logarithmic inequality (New).**
- **Absorption argument.**

- Write
  $f(u_n) = f(v_n + w_n) = f(v_n) + f'(v_n)\, w_n + \frac{1}{2}\, f''(v_n + \theta_n\, w_n)\, w_n^2$.

- Convergence in measure to 0 of (v_n).

- The assumption $\limsup_{n\to\infty}\, \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}$ together with the logarithmic inequality implies that $(f(v_n))$ is bounded in $L^{1+\epsilon}([0,T], L^{2+\epsilon}(\mathbb{R}^2))$ for some $\epsilon > 0$.

- As a consequence

$$\|f(v_n)\|_{L^1([0,T];L^2(\mathbb{R}^2))} \to 0\ .$$

- By Hölder inequality

$$\|f'(v_n)\, w_n\|_{L^1([0,T];L^2(\mathbb{R}^2))} \leq \varepsilon_n \, \|w_n\|_{ST(I)} \quad (\varepsilon_n \to 0) \quad .$$

- For the last term we have

$$\|f''(v_n + \theta_n\, w_n)\, w_n^2\|_{L^1([0,T];L^2(\mathbb{R}^2))} \leq \varepsilon_n \, \|w_n\|_{ST(I)}^2 \quad (\varepsilon_n \to 0),$$

provided that

$$\limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];H^1)} \leq \frac{1 - L\sqrt{4\pi}}{2}.$$

- We conclude by absorption and continuity arguments.

Concluding remarks

- A similar result can be obtained in higher dimensions (work in progress).
- The description of the lack of compactness of the embedding of $H^1(\mathbb{R}^2)$ into Orlicz space in the general frame is much harder than the radial setting (work in progress).
- An interesting (and difficult) question is to remove the radial symmetry assumption in the Struwe's result.

Thank You

Some Publications

- **S. Ibrahim and M. Majdoub**, *Comparaison des ondes linéaires et non-linéaires à coefficients variables*, Bull. Belg. Math. Soc. **10**(2003), 299–312.
- **S. Ibrahim, M. Majdoub and N. Masmoudi**, *Ill-posedness of H^1-supercritical waves*, C. R. Math. Acad. Sci. Paris, **345** (2007), 133–138.
- **S. Ibrahim, M. Majdoub and N. Masmoudi**, *Double logarithmic inequality with a sharp constant*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 87–97.
- **S. Ibrahim, M. Majdoub and N. Masmoudi**, *Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity*, Comm. Pure Appl. Math. **59** (2006), no. 11, 1639–1658.
- **S. Ibrahim, M. Majdoub, N. Masmoudi and K. Nakanishi**, *Scattering for the two-dimensional energy-critical wave equation*, Duke Mathematical Journal, Vol. **150**, 287–329, 2009.
- **M. Majdoub**, *Qualitative study of the critical wave equation with a subcritical perturbation*, J. Math. Anal. Appl., **301** (2005), 354–365.