Isoperimetric inequalities and variations on Schwarz's Lemma

joint work with M. van den Berg and T. Carroll

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Outline

Classical Schwarz Lemma

We begin with the classical form of Schwarz's Lemma. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in either case implies $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.

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We can rephrase this more geometrically by defining

$$\mathrm{Rad}(r) = \sup_{|z|=r} |f(z) - f(0)|.$$

Then Schwarz's Lemma says $\mathrm{Rad}(r) \leq r$, and, in fact, its classical proof implies

$$r \mapsto \frac{1}{r}\,\mathrm{Rad}(r)$$

is an increasing function for $0 < r < 1$.

Variations on Schwarz's Lemma

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Theorem

(van den Berg, Carroll, –) Let $f : \mathbb{D} \to \mathbb{C}$ *be conformal. Then the function*

$$r \mapsto \Phi_\lambda(r) = \frac{\lambda(f(r\mathbb{D}))}{\lambda(r\mathbb{D})} = \frac{1}{j_0^2} r^2 \lambda(f(r\mathbb{D})) \qquad (1)$$

is strictly decreasing for $0 < r < 1$*, unless* f *is linear (in which case this function is constant).*

Outline

A key tool we use is the following isoperimetric inequality.

Theorem

Let $D \subset \mathbb{C}$ be a bounded domain with Lipschitz boundary, and let ϕ be the first Dirichlet eigenfunction. Place the conformal metric $ds^2 = |\nabla\phi|^2|dz|^2$ on D, and let

$$A = \int_D |\nabla\phi|^2|dz|^2, \qquad L = \int_{\partial D} |\nabla\phi||dz|$$

be the area and perimeter (respectively) of D with respect to this conformal metric. Then

$$L^2 \geq 4\pi A, \tag{2}$$

with equality if and only if D is a round disk.

The metric $|\nabla\phi|^2|dz|^2$ is a singular metric on D, with singularities at the critical points of the eigenfunction ϕ. However, there are only finitely many of these singular points, all of which lie in the interior of D, and the metric vanishes to first order there.

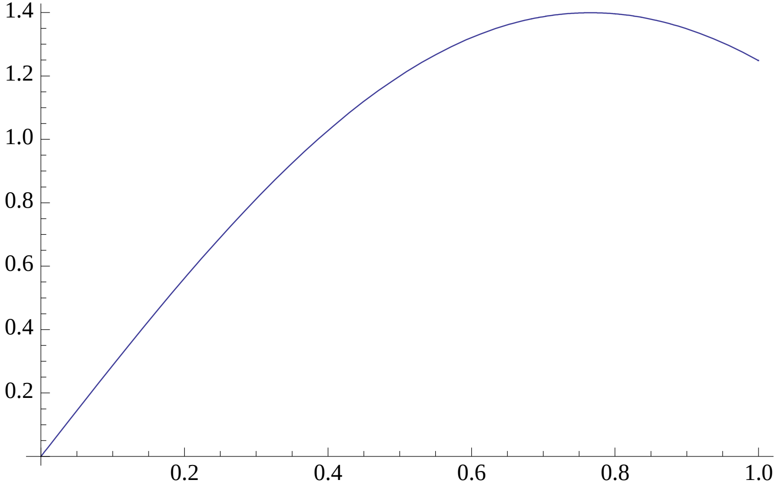
The case of equality: the Bessel disk

On the unit disk $\mathbb{D}$, let $\phi_0(r) = J_0(j_0 r)$ be the first eigenfunction. We call the conformal metric

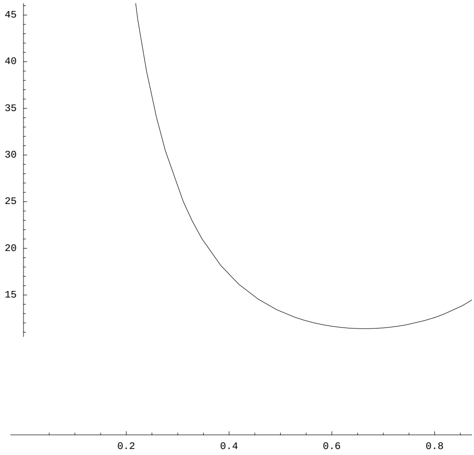
$$ds^2 = |\nabla \phi_0|^2 |dz|^2 = j_0^2 J_1^2(j_0 r)|dz|^2$$

the Bessel metric on the disk. This metric has an isolated singularity at the origin. The curvature is everywhere positive, and goes to $+\infty$ as $r \to 0^+$, and $(\mathbb{D}, |\nabla \phi_0|^2 |dz|^2)$ has total curvature 4π.

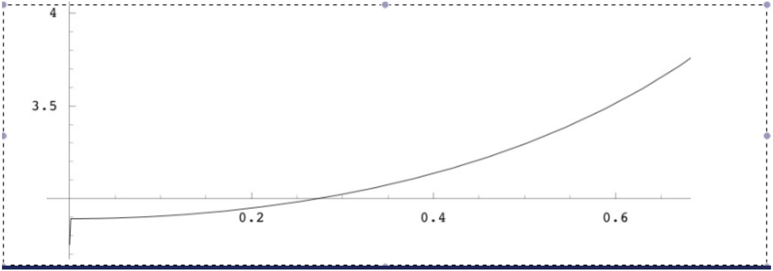
This is a graph of the conformal factor $\rho(r) = j_0 J_1(j_0 r)$:



This is a graph of the Gauss curvature, $K = -\rho^{-2}\Delta\log(\rho)$, of the Bessel disk:



This is a graph of the Gauss-Bonnet integrand, $KdA = -\Delta \log(\rho)$, of the Bessel disk:



There's a fairly large literature of isoperimetric inequalities on surfaces. Notable recent results include a pair of theorems due (separately) to Topping and Morgan-Hutchings-Howard. In each of their inequalities, they note that one gets equality for a rotationally symmetric on a disk where the curvature function is monotone.

We note that the Bessel disk achieves equality in their inequalities, is rotationally symmetric, but its curvature is not monotone.

Outline

Schwarz's Lemma and variations

Isoperimetric inequalities

Proof

We'll briefly sketch the proof of (2) and how one proves (1) using (2).
Payne and Rayner proved an equivalent form of this isoperimetric inequality for the first eigenfunction:

$$\left(\int_D \phi dA\right)^2 \geq \frac{4\pi}{\lambda}\int_D \phi^2 dA. \tag{3}$$

Their proof relies on the coarea formula and the Cauchy-Schwarz inequality, and is fairly similar to the classical proof of the Faber-Krahn inequality.

It is straightforward to see that (2) and (3) are equivalent. If η is the outward unit normal of D, then

$$\begin{aligned} \int_{\partial D} |\nabla \phi| ds &= -\int_{\partial D} \frac{\partial \phi}{\partial \eta} ds \\ &= -\int_D \Delta \phi dA \\ &= \lambda \int_D \phi dA \end{aligned}$$

Also, because ϕ is minimizes the Rayleigh quotient,

$$\int_D |\nabla \phi|^2 dA = \lambda \int_D \phi^2 dA.$$

Now we'll prove (1); recall

$$\Phi_\lambda(r) = \frac{r^2}{j_0^2}\lambda(f(r\mathbb{D})).$$

To show that Φ_λ is decreasing, we want to show that

$$0 \geq \frac{d\Phi_\lambda}{dr} = \frac{1}{j_0^2}\left[2r\lambda(f(r\mathbb{D})) + r^2\frac{d}{dr}\lambda(f(r\mathbb{D}))\right],$$

or, equivalently,

$$\frac{2}{r}\lambda(f(r\mathbb{D})) \leq -\frac{d}{dr}(\lambda(f(r\mathbb{D}))). \tag{4}$$

A classical variation formula of Hadamard tells us

$$\frac{d}{dr}\lambda(f(r\mathbb{D})) = -r\int_0^{2\pi} |\nabla\psi_r|^2 d\theta, \tag{5}$$

where ϕ_r is the eigenfunction of $D_r = f(r\mathbb{D})$ and $\psi_r = \phi_r \circ f$. We can normalize ϕ_r so that $\int_{D_r} \phi_r^2 dA = 1$.

By (2),

$$\begin{aligned}\left(r\int_0^{2\pi}|\nabla\psi_r||dz|\right)^2 &= \left(\int_{\partial D_r}|\nabla\phi_r|ds\right)^2\\ &\geq 4\pi\int_{D_r}|\nabla\phi_r|^2dA\\ &= 4\pi\int_{r\mathbb{D}}|\nabla\psi_r|^2|dz|^2.\end{aligned}$$

Now, using the normalization of ϕ_r,

$$\begin{aligned}
\frac{2}{r}\lambda(f(r\mathbb{D})) &= \frac{2}{r}\int_{D_r} |\nabla\phi_r|^2 dA = \frac{2}{r}\int_{r\mathbb{D}} |\nabla\psi_r|^2 |dz|^2 \\
&\leq \frac{r}{2\pi}\left(\int_0^{2\pi} |\psi_r(re^{i\theta})| d\theta\right)^2 \\
&\leq r\int_0^{2\pi} |\nabla\psi_r(re^{i\theta})|^2 d\theta = -\frac{d}{dr}\lambda(f(r\mathbb{D})).
\end{aligned}$$

Thanks!