Isoperimetric Inequalities and Variations on Schwarz's Lemma

Tom Carroll and Jesse Ratzkin

University College Cork and University of Cape Town

t.carroll@ucc.ie and j.ratzkin@ucc.ie

June 10, 2010

Abstract

In this note we prove a version of the classical Schwarz lemma for the first eigenvalue of the Laplacian with Dirichlet boundary data. A key ingredient in our proof is an isoperimetric inequality for the first eigenfunction, due to Payne and Rayner, which we reinterpret as an isoperimetric inequality for a (singular) conformal metric on a bounded domain in the plane.

1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disk in the complex plane, and let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$. Then the classical Schwarz lemma states that $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$, and that equality in either case implies $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. One can reinterpret this result more geometrically by defining, for $0 < r < 1$,

$$\mathrm{Rad}(r) = \sup_{|z|=r} |f(z) - f(0)|,$$

so that the Schwarz lemma states $\mathrm{Rad}(r) \leq r\,\mathrm{Rad}(1)$ for every analytic $f : \mathbb{D} \to \mathbb{C}$. In fact, the classical proof of the Schwarz lemma implies

$$\Phi_{\mathrm{Rad}}(r) = \frac{\mathrm{Rad}(r)}{r}$$

is a strictly increasing function of r, unless f is linear (in which case Φ_{Rad} is constant).

Burckel, Marshall, Minda, Poggi-Corradini, and Ransford [1] recently proved versions of the Schwarz Lemma for diameter, logarithmic capacity, and area. They asked whether similar inequalities hold for other quantities, such as the first eigenvalue λ of the Laplacian with Dirichlet boundary data.

Theorem 1. *Let f be a conformal mapping of the unit disk $\mathbb{D}$. The function*

$$\Phi_\lambda(r) = \frac{\lambda\big(f(r\mathbb{D})\big)}{\lambda(r\mathbb{D})} = \frac{1}{j_0^2} r^2 \lambda\big(f(r\mathbb{D})\big), \quad 0 < r < 1, \tag{1}$$

is strictly decreasing, unless f is linear (in which case Φ_λ is constant).

Taking a limit of the right hand side of (1) as $r \to 0^+$, we recover the estimate in Section 5.8 of [6]. Using monotonicity of the right hand side of (1), we also see that the limit as $r \to 1^-$ of the right hand side of (1) also exists, thought it might be zero.

A slight modification of the proof of Theorem 1 yields the following corollary.

Corollary 2. *Let f be an analytic function in the unit disk. For $0 < r < 1$ let Σ_r be the Riemann surface associated to $f : r\mathbb{D} \to \mathbb{C}$. Then the function*

$$r \mapsto \frac{1}{j_0^2} r^2 \lambda(\Sigma_r)$$

is strictly decreasing, unless f is linear (in which case this function is constant).

Remark 1. *After presenting these results at the Queen Dido conference on isoperimetry, we learned that Laugesen and Morpurgo [4] proved a series of very general results which includes the inequality of Theorem 1. (See, in particular, Theorem 7, on page 80 of their paper.) Our proof is quite different from that in [4] and may be of interest in its own right.*

Remark 2. *One key point of [1] is that their estimates involve the area (for instance) of the image of $f(r\mathbb{D})$, rather than the area with multiplicity. We have left the corresponding question for the first Dirichlet eigenvalue of the Laplacian open. In this case, the first variation formula for the first eigenvalue of the image domain $f(r\mathbb{D})$ is more complicated than the variation formula we have below, and when pulled back to $r\mathbb{D}$ will involve an integral over a proper subset of the boundary circle.*

A key step in the proof of this eigenvalue estimate is to rewrite a result of Payne and Rayner [7] as an isoperimetric-type inequality for the first eigenfunction.

Theorem A. *Let D be a bounded planar region with Lipschitz boundary ∂D, and let ϕ be the first eigenfunction of the Laplacian with Dirichlet boundary conditions. Then*

$$\left(\int_{\partial D} |\nabla \phi| \right)^2 \geq 4\pi \int_D |\nabla \phi|^2, \tag{2}$$

with equality if and only if D is a disk.

The inequality (2) is in fact the isoperimetric inequality $L^2 \geq 4\pi A$ for the domain D, where one measures length and area with respect to the (singular) conformal metric $ds^2 = |\nabla \phi|^2 |dw|^2$. We discuss some properties of this metric below, in Section 2, and the equality in the case of the disk in Section 3.

The rest of this paper proceeds as follows. We prove Theorem 1 in Section 2 by writing out the first variation of the eigenvalue under a domain perturbation and reducing our problem to the isoperimetric inequality in Theorem A. We examine the equality case of the isoperimetic inequality, that of a disk, in Section 3.

Acknowledgements: We first learned about these variations on Schwarz's Lemma from Pietro Poggi-Corradini during a *Summer School in Conformal Geometry, Potential Theory, and Applications* at NUI Maynooth in June 2009. We would like to thank Poggi-Corradini for interesting discussions on the subject and the organizers of the conference for providing a stimulating venue for these discussions. We would also like to thank Michiel van den Berg for many enlightening conversations, and telling us of [7]. Finally, we would like to thank Rick Laugesen for pointing out [4].

2 A Schwarz Lemma for the first eigenvalue

Let f be an analytic function in the unit disk $\mathbb{D}$. Let $D_r = f(r\mathbb{D})$ for $0 < r < 1$, and let $\lambda(r) = \lambda(D_r)$. The first Dirichlet eigenvalue for a disk is known to be $\lambda(r\mathbb{D}) = j_0^2/r^2$ where j_0 is the first positive zero of the Bessel function J_0 of index zero. We then have

$$\Phi(r) = \frac{\lambda\big(f(r\mathbb{D})\big)}{\lambda(r\mathbb{D})} = \frac{1}{j_0^2}\, r^2 \lambda(r). \tag{3}$$

Taking a derivative, we see that

$$\frac{d\Phi}{dr} = \frac{1}{j_0^2}\left[2r\lambda(r) + r^2 \frac{d\lambda}{dr}\right], \tag{4}$$

so Φ is a decreasing function of r precisely if

$$\frac{2}{r}\,\lambda(r) \le -\frac{d\lambda}{dr}. \tag{5}$$

A classical theorem of Hadamard [3] computes the first variation of the eigenvalue as follows (see also [8, 2, 6]).

Let Ω_0 be a domain. Let $\zeta(t,x)$ be a flow on Ω_0 associated with the variation field $\chi = \chi(t,x)$ in the time interval $(-t_0, t_0)$, in that,

$$\frac{\partial \zeta}{\partial t}(t,x) = \chi\big(\zeta(t,x)\big) \tag{6}$$

$$\zeta(0,p) = p, \quad p \in \Omega_0. \tag{7}$$

Let Ω_t be the domain $\zeta(t,\Omega_0)$, let $\lambda(t)$ be the first Dirichlet eigenvalue for the Laplacian in Ω_t, and let $\phi(t,x)$, $x \in \Omega_t$, be the associated eigenfunction normalised so that $\int_{\Omega_t} \phi^2 = 1$. Let η denote the outward normal and $d\sigma$ denote arc-length measure for $\partial\Omega_t$. For the reader's convenience, we include a proof of the following formula for the time derivative of the eigenvalue, which draws heavily on the treatment in [6]. We take all boundaries and variation fields to be C^∞, even though the variation formula holds with less regularity. In the calculation below we denote differentiation with respect to the parameter t with a dot.

Lemma 3.

$$\dot{\lambda}(0) = -\int_{\partial\Omega_0} \langle \chi, \eta\rangle \left(\frac{\partial\phi}{\partial\eta}\right)^2 d\sigma. \tag{8}$$

Remark 3. *Because the first eigenvalue is simple, the function $\lambda(t)$ is differentiable. The higher eigenvalues $\lambda_k(t)$, for $k > 1$, may not be differentiable functions of t, but both one-sided derivatives will exist. See the discussion in Sections 2 and 3 of [2] for more information.*

Proof. First we compute the time derivative of the boundary terms of the normalized first eigenfunction ϕ. Taking a derivative of the condition

$$\phi\big(t, \zeta(t,p)\big) = 0,\ p \in \partial\Omega_0$$

with respect to t and using (6), we obtain

$$\dot{\phi}\big(t,\zeta(t,p)\big) + \big\langle \nabla\phi\big(t,\zeta(t,p)\big), \chi(p)\big\rangle = 0.$$

Here and later, the gradient refers only to the spatial derivative. Set $t = 0$ and use the fact that ϕ is constant along $\partial\Omega_t$ to obtain

$$\dot{\phi}(0,p) = -\big\langle \nabla\phi(0,p), \chi(p)\big\rangle = -\Big\langle \frac{\partial\phi}{\partial\eta}\Big|_{(0,p)} \eta(p), \chi(p)\Big\rangle, \quad p \in \partial\Omega_0. \tag{9}$$

Next we take the derivative of the eigenfunction equation

$$\Delta\phi\big(t,\zeta(t,p)\big)+\lambda(t)\phi\big(t,\zeta(t,p)\big)=0 \tag{10}$$

with respect to t and evaluate at $t=0$. This leads to

$$\begin{aligned}0 &= \Delta\left[\dot\phi+\langle\nabla\phi,\chi\rangle\right]+\lambda(t)\left[\dot\phi+\langle\nabla\phi,\chi\rangle\right]+\dot\lambda(t)\phi\\ &= \Delta\dot\phi+\langle\nabla\Delta\phi,\chi\rangle+\lambda(t)\dot\phi+\lambda(t)\langle\nabla\phi,\chi\rangle+\dot\lambda(t)\phi\\ &= \Delta\dot\phi+\lambda(t)\phi_t+\dot\lambda(t)\phi.\end{aligned}$$

Setting $t=0$ and rearranging yields

$$\Delta\ \dot\phi\Big|_{t=0}+\lambda(0)\ \dot\phi\Big|_{t=0}=-\dot\lambda(0)\phi\big|_{t=0}\quad\text{ in }\Omega_0. \tag{11}$$

We multiply (10), with $t=0$, by $\dot\phi\big|_{t=0}$ and multiply (11) by ϕ, subtract and obtain

$$\dot\lambda(0)\phi^2(0,p)=\dot\phi(0,p)\Delta\phi(0,p)-\phi(0,p)\Delta\dot\phi(0,p),\quad p\in\Omega_0. \tag{12}$$

Integrate (12) over Ω_0 and use the fact that $\int_{\Omega_t}\phi^2=1$ to obtain

$$\begin{aligned}\dot\lambda(0) &= \int_{\Omega_0}\dot\phi\Delta\phi-\phi\Delta\dot\phi\\ &= \int_{\partial\Omega_0}\dot\phi\frac{\partial\phi}{\partial\eta}-\int_{\Omega_0}\langle\nabla\phi,\nabla\dot\phi\rangle+\int_{\Omega_0}\langle\nabla\phi,\nabla\dot\phi\rangle-\int_{\partial\Omega_0}\phi\frac{\partial\dot\phi}{\partial\eta}\\ &= \int_{\partial\Omega_0}\dot\phi\frac{\partial\phi}{\partial\eta}\\ &= -\int_{\partial\Omega_0}\frac{\partial\phi}{\partial\eta}\langle\nabla\phi,\chi\rangle\\ &= -\int_{\partial\Omega_0}\langle\chi,\eta\rangle\left(\frac{\partial\phi}{\partial\eta}\right)^2,\end{aligned}$$

which is equation (14) as claimed. In the second equality above we integrated by parts, in the next to last we used (9), and at the last step we used the fact that ϕ is constant on $\partial\Omega_0$ (and hence $\nabla\phi=\frac{\partial\phi}{\partial\eta}\eta$ there). □

We adapt this formula to our particular case.

Lemma 4. *Let f be a conformal mapping of the unit disk $\mathbb{D}$ with $f(0)=0$. Let $\lambda(r)$ be the eigenvalue of the domain $D_r=f(r\mathbb{D})$ with eigenfunction ϕ_r in $L^2(D_r)$. Let*

$$\psi_r(z)=\phi_r\big(f(z)\big),\quad z\in r\mathbb{D}. \tag{13}$$

Then

$$\frac{d\lambda}{dr}=-r\int_0^{2\pi}|(\nabla\psi_r)(re^{i\theta})|^2\,d\theta. \tag{14}$$

Remark 4. *The function ψ satisfies the equation*

$$\Delta\psi+\lambda|f'|^2\psi=0,$$

and so ψ is the first Dirichlet eigenfunction of the Laplacian on the conformal disk $(\mathbb{D},|f'|^2|dz|^2)$.

Proof. For a fixed r in $(0,1)$, we set

$$\zeta(t,p) = f\big((1+t/r)f^{-1}(p)\big), \quad p \in D_r. \tag{15}$$

Then, $\zeta(0,p) = p$, $\zeta(0,D_r) = \Omega_0 = D_r$, $\zeta(t,D_r) = \Omega_t = D_{r+t}$, and

$$\frac{\partial \zeta}{\partial t}(t,p) = f'\big((1+t/r)f^{-1}(p)\big)\frac{1}{r}f^{-1}(p).$$

It follows that (6) holds with

$$\chi(\zeta) = \frac{1}{r+t}f^{-1}(\zeta)f'\big(f^{-1}(\zeta)\big). \tag{16}$$

The unit normal vector to the boundary of D_r at ζ is

$$\eta(\zeta) = \frac{f^{-1}(\zeta)}{r}\,\frac{f'\big(f^{-1}(\zeta)\big)}{\big|f'\big(f^{-1}(\zeta)\big)\big|}. \tag{17}$$

Thus, (16) with $t=0$ and (17) show that $\chi(\zeta) = \big|f'\big(f^{-1}(\zeta)\big)\big|\eta(\zeta)$, $\zeta \in \partial D_r$, so that

$$\langle \chi, \eta \rangle = \big|f'\big(f^{-1}(\zeta)\big)\big|, \quad \zeta \in \partial D_r.$$

The gradient of the function ψ_r given by (13) is $|\nabla\psi_r(z)| = |\nabla\phi_r\big(f(z)\big)|\,|f'(z)|$. This, and Lemma 3, lead to

$$\begin{aligned}
\dot\lambda(r) &= -\int_{\partial D_r} \big|f'\big(f^{-1}(\zeta)\big)\big|\,|\nabla\phi_r(\zeta)|^2\,|d\zeta| \\
&= -\int_{C(0,r)} |f'(z)|\left(\frac{|\nabla\psi_r(z)|}{|f'(z)|}\right)^2 |f'(z)|\,|dz| \\
&= -\int_{C(0,r)} |\nabla\psi_r(z)|^2\,|dz|,
\end{aligned}$$

where $C(0,r)$ denotes the circle centre 0 and radius r, which is (14). $\square$

Next we verify Theorem A. Payne and Rayner write the inequality in the form

$$\left(\int_D \phi\right)^2 \geq \frac{4\pi}{\lambda}\int_D \phi^2 \tag{18}$$

with equality if and only if D is a disk. If we denote by η the unit outward normal to the boundary of D,

$$\int_{\partial D} |\nabla\phi| = \int_{\partial D}\left(-\frac{\partial\phi}{\partial\eta}\right) = \int_D (-\Delta\phi) = \lambda\int_D \phi,$$

where the first equality comes from the fact that ϕ is constant on the boundary of D, the second from Green's theorem, and the third from the eigenfunction equation $\Delta\phi + \lambda\phi = 0$. Since ϕ minimises the Rayleigh quotient,

$$\int_D |\nabla\phi|^2 = \lambda\int_D \phi^2.$$

Hence, (2) and (18) are equivalent.

Proof of Theorem 1. By Theorem A,

$$\left(\int_{C(0,r)} \big|\nabla\psi_r(z)\big|\,|dz|\right)^2 = \left(\int_{\partial D_r)} |\nabla\phi_r|\right)^2 \geq 4\pi\int_{D_r} |\nabla\phi_r|^2 = 4\pi\int_{r\mathbb{D}} |\nabla\psi_r|^2.$$

Hence, since $\|\phi_r\|_{L^2(D_r)} = 1$,

$$\begin{aligned}\frac{2}{r}\,\lambda(r) = \frac{2}{r}\int_{D_r} |\nabla\phi_r|^2 = \frac{2}{r}\int_{r\mathbb{D}} |\nabla\psi_r|^2 \\ &\leq \frac{r}{2\pi}\left(\int_0^{2\pi} |\nabla\psi_r(re^{i\theta})|\,d\theta\right)^2 \\ &\leq r\int_0^{2\pi} |\nabla\psi_r(re^{i\theta})|^2\,d\theta \\ &= -\lambda'(r),\end{aligned}$$

by (14). This proves (5) and hence Theorem 1. □

Remark 5. *The metric $|\nabla\phi|^2|dw|^2$ on $D_r = f(r\mathbb{D})$ is a singular metric, with singularities at the critical points of the eigenfunction ϕ. However, ϕ solves a second order, linear, elliptic equation on a bounded domain with C^∞ boundary, so it only has finitely many critical points, none of which are degenerate. We conclude that the metric $|\nabla\phi|^2|dw|^2$ in D_r, or, equivalently, $|\nabla\psi|^2|dz|^2$ in $r\mathbb{D}$, has only finitely many singular points, where the metric vanishes only to zeroth order.*

Remark 6. *The isoperimetric inequality $L^2 \geq 4\pi A$, with $L = \int_{\partial D} |\nabla\phi||dw|$ and $A = \int_D |\nabla\phi|^2|dw|^2$, seems quite general. Using the Riemann mapping theorem, it holds for any simply connected domain $D \subset \mathbb{C}$. We also find it remarkable that the constant in this isoperimetric inequality is the same one as in the classical isoperimetric inequality.*

Proof of Corollary 2. The function $f : r\mathbb{D} \to \Sigma_r$ is a conformal map away from its critical points, and so the first variation formula (14) holds so long as f does not have a critical point of length r. For any $r_0 \in (0,1)$ there are only finitely many values $\hat{r} < r_0$ such that f has a critical point of length equal to $\hat{r}$, and the variation formula (14) is valid away from these values $\hat{r}$. Thus, we can integrate the inequality (5) to see that $\Phi(r) = (r^2/j_0^2)\lambda(f(r\mathbb{D}))$ is decreasing for $0 < r < r_0$. Moreover, if there exist $r_1 < r_2$ such that $\Phi(r_1) = \Phi(r_2)$ then f is linear on the annulus $r_1 < |z| < r_2$; this combined with the fact that f is analytic on the disk $\mathbb{D}$ implies f is linear on the whole disk. □

3 The Bessel disk: the equality case of the isoperimetric inequality

In any given inequality, the case of equality is always important, and often sheds light on other problems. The equality case of (1) occurs when f is linear, in which case the image domains $f(r\mathbb{D})$ are disks for all $r \in (0,1)$. In this case, the eigenfunctions ϕ and ψ agree up to scaling factors, and we write $\phi(z) = J_0(j_0|z|)$, where J_0 is the Bessel function with index zero and j_0 is its first positive root.

Definition 1. *We call the unit disc $\mathbb{D}$ equipped with the conformal metric*

$$ds = J_1(j_0|z|)|dz|$$

the Bessel disc.

The following lemma is an immediate consequence of (1).

Lemma 5. *In the Bessel disc, $L^2 = 4\pi A$.*

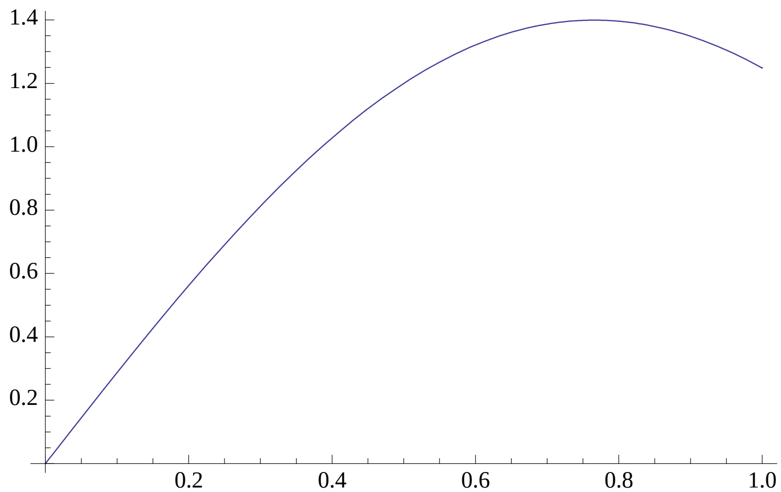

Figure 1: This is a plot of the conformal factor $\rho(r) = j_0 J_1(j_0 r)$ for the Bessel disk.

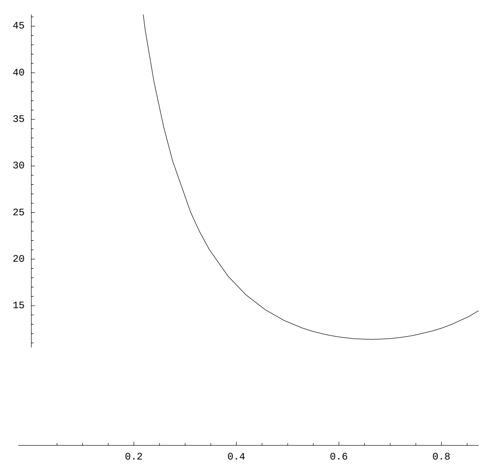

Figure 2: This is a plot of the curvature $K = -\rho^{-2}\Delta \log(\rho)$ for the Bessel disk.

We conclude this section by exploring the geometry of the Bessel disk. It is convenient to recall the formula for the Gauss curvature of a conformal metric. In general the metric $ds = \rho|dz|$ has Gauss curvature

$$K = -\frac{1}{\rho^2}\Delta(\log \rho).$$

In particular, negative curvature is equivalent to $\log \rho$ being subharmonic.

Observe that the Bessel disk has positive curvature, which blows up logarithmically at the origin. We include plots of the curvature and the Gauss-Bonnet integrand for the reader's enlightenment.

Lemma 6. *The total curvature of the Bessel disc is 4π.*

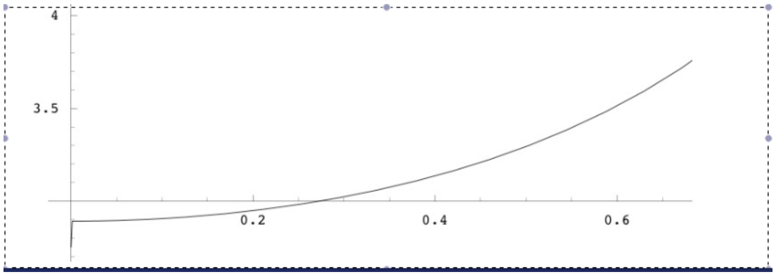

Figure 3: This is a plot of the Gauss-Bonnet integrand $KdA = -\Delta\log(\rho)|dz|^2$ for the Bessel disk.

Proof. Let $\rho(z) = J_1(j_0|z|)$, so that

$$\begin{aligned}\int_{\mathbb{D}} KdA &= -\int_{\mathbb{D}} \frac{\Delta\log\rho}{\rho^2}\rho^2|dz|^2 = -2\pi\int_0^1 [(\log\rho)''(r) + r^{-1}(\log\rho)'(r)]rdr \\ &= -2\pi\int_0^1 r(\log\rho)''(r) + (\log\rho)'(r)dr = -2\pi\int_0^1 \frac{d}{dr}(r(\log\rho)')dr \\ &= 2\pi\left(\frac{r\rho'(r)}{\rho(r)}\right)\Bigg|_0^1 = 2\pi\left(\lim_{r\to 0}\frac{rj_0J_1'(j_0r)}{J_1(j_0r)} - \frac{j_0J_1'(j_0)}{J_1(j_0)}\right) \\ &= 4\pi.\end{aligned}$$

Here we have used Bessel identities to show $j_0J_1'(j_0) = -J_1(j_0)$. □

Looking closely at this computation, we see that the Gauss-Bonnet integrand is an exact derivative, and so there are two terms which contribute to the total curvature: a boundary term and an interior term at the critical point of the first eigenfunction. For any bounded domain D with Lipschitz boundary, the local behavior of its first eigenfunction near a critical point will be that of the Bessel function at the origin of the disk, at least to first order. Thus, the computation above shows that any critical point of the first eigenfunction will contribute 2π to the total curvature of $(D, |\nabla\phi|^2|dz|^2)$, where ϕ is the first eigenfunction of D. It therefore seems natural to conjecture that, for instance, the total curvature of $(D, |\nabla\phi|^2|dz|^2)$ is exactly 4π for any convex domain D.

We contrast the Bessel disk with the isoperimetric inequalities of Topping [9] and Morgan-Hutchings-Howard [5]. They prove that a rotationally symmetric metric with a monotone curvature function will achieve equality in each of their inequalities. On the other hand, one can verify the following properties of the Bessel disk by explicit computation. First, it is a rotationally symmetric metric, which realizes equality in both the isoperimetric inequalities of [9] and [5]. Second, the curvature is *n*ot monotone. It remains an interesting open question to characterize which metrics achieve equality in the isoperimetric inequalities of [9] and [5].

References

[1] R. Burckel, D. Marshall, D. Minda, P. Poggi-Corradini, and T. Ransford. *Area, capacity, and diameter versions of Schwarz's lemma.* Conform. Geom. Dyn. **12** (2008), 133–151.

[2] A. El Soufi and S. Ilias. *Domain deformations and eigenvalues of the Dirichlet Laplacian in a Riemannian manifold.* Illinois J. Math. **51** (2007), 645–666.

[3] J. Hadamard. *Mémoire sur le probléme d'analyse relatif á l'équilibre des plaques élastiques encastrées.* Mémoires présentés par divers savants á l'Académie des Sciences. **33** (1908).

[4] R. Laugesen and C. Morpurgo. *Extremals for eigenvalues of Laplacians under conformal mappings.* J. Funct. Anal. **155** (1998), 64–108.

[5] F. Morgan, M. Hutchings, and H. Howards. *The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature.* Tran. Amer. Math. Soc. **352** (2000), 4889–4909.

[6] F. Pacard and P. Sicbaldi. *Extremal domains for the first eigenvalue of the Laplace-Beltrami operator.* Ann. Inst. Fourier **59** (2009), 515–542.

[7] L. Payne and M. Rayner. *An isoperimetric inequality for the first eigenfunction in the fixed membrane problem.* A. Angew. Math. Phys. **23** (1972), 13–15.

[8] M. Schiffer. *Hadamard's formula and variation of domain-functions.* Amer. J. Math. **68** (1946), 417–448.

[9] P. Topping. *The isoperimetric inequality on a surface.* Manuscripta Math. **100** (1999), 23–33.