The method of rotations tight frames: sharp upper bounds on Laplace eigenvalues

(joint work with Richard Laugesen)

Bartłomiej Siudeja

University of Illinois

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Dirichlet eigenvalues

Let λ_i be all solutions of $-\Delta u = \lambda u$, where Δ is the Dirichlet Laplacian on a domain Ω. Then

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty.$$

Neumann eigenvalues

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$$0 = \mu_1 < \mu_2 \le \mu_3 \le \mu_4 \le \cdots \to \infty.$$

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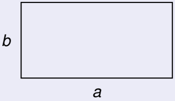
Exact formulas for Dirichlet eigenvalues of rectangles

$$\lambda_{n,m} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right), \; m, n \geq 1.$$



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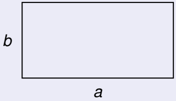

Some examples for $\lambda_{1,1} A (= \lambda_1 A)$

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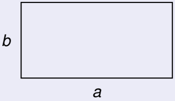

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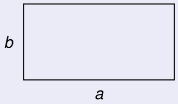

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Theorem (Pólya)

Among all quadrilaterals, squares minimize $\lambda_1 A = \lambda_{1,1} A$.

Some examples for $\lambda_1 \frac{A^2}{L^2}$ (L - perimeter)

$$a = 2, b = 1 : \qquad \frac{250}{1800}\pi^2,$$
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Theorem (Hersch, 1966)

Among parallelograms, squares maximize $\lambda_1 \frac{A^3}{I}$.

Theorem (Freitas, 2006)

Among triangles, equilateral triangles maximize $\lambda_1 \frac{A^3}{I}$.

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Theorem (Pólya, 1952, Pólya & Schiffer, 1954)

Start with an n-fold rotationally symmetric domain. Among all domains obtained by stretching, the original domain maximizes $\lambda_1 \frac{A^3}{I}$.

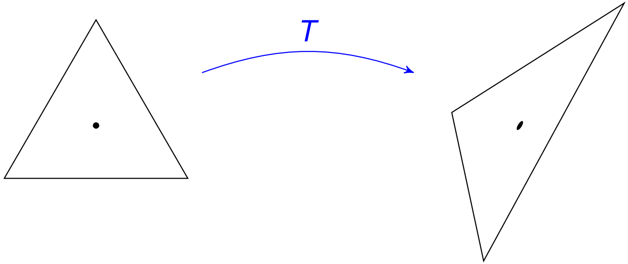
This result follows from rotational symmetry of the first eigenfunction.

Theorem (Laugesen & Siudeja, 2010)

Suppose D has N-fold rotational symmetry of order $N \geq 3$ and T be a linear transformation. Then

$$(\lambda_1 + \cdots + \lambda_n)|_{T(D)} \leq \frac{1}{2}\|T^{-1}\|_{HS}^2 (\lambda_1 + \cdots + \lambda_n)|_D,$$

$\|M\|_{HS}^2 = \sum M_{ij}^2$ *(Hilbert-Schmidt norm). Equality holds if $T = Id$.*



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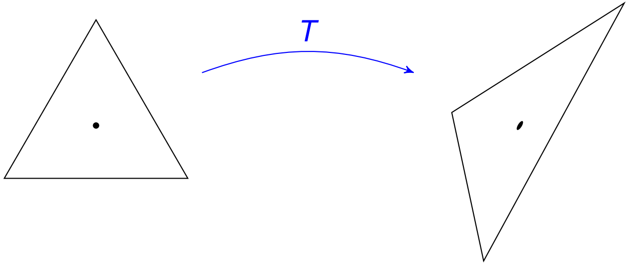

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The proof is very flexible

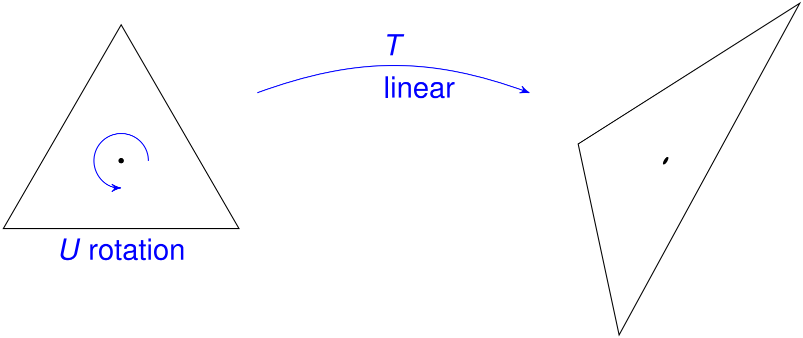
Similar results can be obtained for

- Robin boundary conditions,
- certain Schrödinger operators,
- some nonlocal operators.

Method of Rotations and Tight Frames

Let u_i be orthonormal eigenfunctions on D and U_m a rotation by $2\pi m/N$. By taking a trial functions $u_i \circ U_m \circ T^{-1}$ on $T(D)$

$$\sum_i \lambda_i(T(D)) \leq \sum_i \int_D |(\nabla u_i) U_m T^{-1}|^2 dx = \sum_i \sum_k \int_D |(\nabla u_i) U_m T_k^{-1}|^2 dx$$



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We average over $m = 1, \ldots, N$ using Plancherel identity for rotational orbit $\left\{U_1 \vec{t}, \ldots, U_N \vec{t}\right\}$ (Tight Frame)

$$\frac{1}{N} \sum_{m=1}^{N} |\vec{s} \cdot (U_m \vec{t})|^2 = \frac{1}{2} |\vec{s}|^2 |\vec{t}|^2,$$

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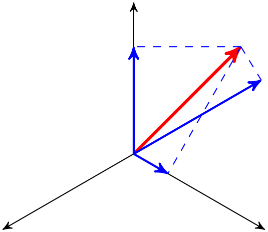
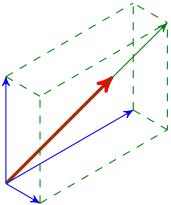
$$\sum_i \lambda_i(T(D)) \le \sum_i \sum_k \int_D |\nabla u_i|^2 |T_k^{-1}|^2 dx = \frac{\|T^{-1}\|_{HS}^2}{2} \sum_i \int_D |\nabla u_i|^2 dx$$

Mercedes Tight Frame: N=3.





$$\sum_{m=1}^{3} |\vec{s} \cdot (U_m \vec{t})|^2 = \frac{3}{2} |\vec{s}|^2 |\vec{t}|^2.$$

Corollary (Case $n = 1$ proved by Pólya, 52)

$$(\lambda_1 + \cdots + \lambda_n) A \frac{A^2}{I} \quad \textit{maximal for} \quad \begin{cases} \textit{equilateral among triangles} \\ \textit{square among parallelograms} \\ \textit{disk among ellipses} \end{cases}$$

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Proof

Taking trace of moment matrix of $T(D)$ gives

$$\frac{1}{2}\|T^{-1}\|_{HS}^2 = \frac{I}{A^3}(T(D)) \Big/ \frac{I}{A^3}(D).$$

Higher dimensions

Theorem (Laugesen & Siudeja, 2010)

Let D be a d-dimensional tetrahedron, or a cube, or a ball. (Then $T(D)$ is a simplex, or a parallelepiped, or an ellipsoid.) We have

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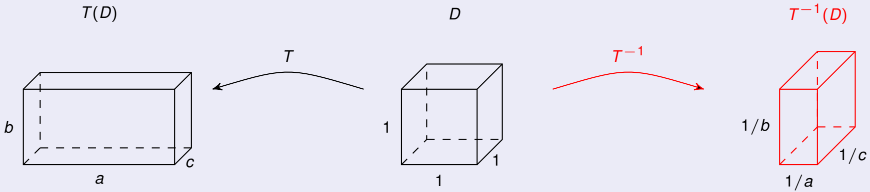
Corollary

$$(\lambda_1 + \cdots + \lambda_n)V^{2/d}\Big|_{T(D)} \cdot \frac{V^{1+2/d}}{I}\Big|_{T^{-1}(D)}$$ *maximal for $T =$Identity*

$$\lambda_1 V^{2/3}\Big|_{T(D)} \cdot \frac{V^{1+2/3}}{I}\Big|_{T^{-1}(D)} \qquad \text{maximal for } T = \text{Identity}$$

Intuition for using $T^{-1}(D)$



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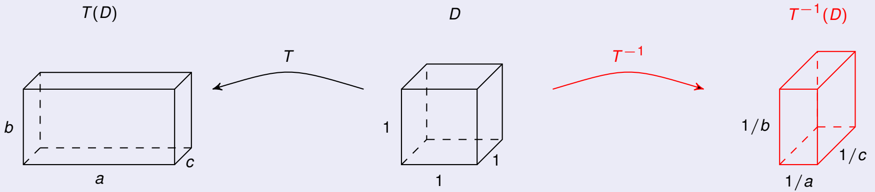

$$\lambda_1 = \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \qquad I = \frac{V}{12}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

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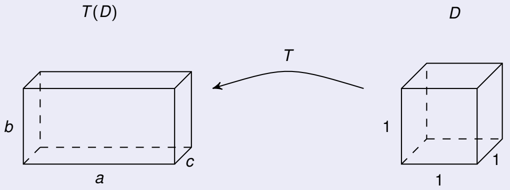

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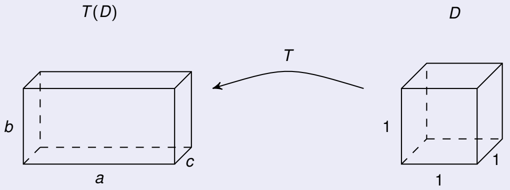

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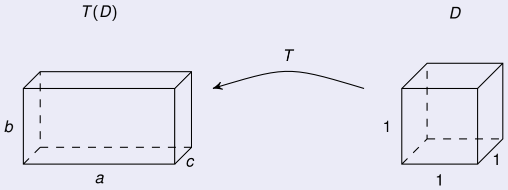

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For any rotationally symmetric domains even if eigenvalues are unknown.

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Explicit extremal domains.

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All eigenvalue sums for all boundary conditions.

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In particular we could use a domain obtained from a cube by adding a rotationally invariant variation to each face.

General convex domains

- To get an upper bound with disk as a maximizer we cannot evaluate the roundness controlling factor on the original domain.
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