Calculus of Variations with Fractional and Classical Derivatives

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Abstract: We give a proper fractional extension of the classical calculus of variations. Necessary optimality conditions of Euler-Lagrange type for variational problems containing both fractional and classical derivatives are proved. The fundamental problem of the calculus of variations with mixed integer and fractional order derivatives as well as isoperimetric problems are considered.

Keywords: variational analysis; optimality; Riemann-Liouville fractional operators; fractional differentiation; isoperimetric problems.

1. INTRODUCTION

One of the classical problems of mathematics consists in finding a closed plane curve of a given length that encloses the greatest area: the *isoperimetric problem*. The legend says that the first person who solved the isoperimetric problem was Dido, the queen of Carthage, who was offered as much land as she could surround with the skin of a bull. Dido's problem is nowadays part of the *calculus of variations* [Gelfand and Fomin, 1963, van Brunt, 2004].

Fractional calculus is a generalization of (integer) differential calculus, allowing to define derivatives (and integrals) of real or complex order [Kilbas et al., 2006, Miller and Ross, 1993, Podlubny, 1999]. The first application of fractional calculus belongs to Niels Henrik Abel (1802–1829) and goes back to 1823 [Abel, 1965]. Abel applied the fractional calculus to the solution of an integral equation which arises in the formulation of the *tautochrone problem*. This problem, sometimes also called the *isochrone problem*, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead placed on the wire slides to the lowest point of the wire in the *same time* regardless of where the bead is placed. The cycloid is the isochrone as well as the *brachistochrone* curve: it gives the *shortest time* of slide and marks the born of the *calculus of variations*.

The study of fractional problems of the calculus of variations and respective Euler-Lagrange type equations is a subject of current strong research due to its many applications in science and engineering, including mechanics, chemistry, biology, economics, and control theory. In 1996-1997 Riewe obtained a version of the Euler–Lagrange equations for fractional variational problems combining the conservative and nonconservative cases [Riewe, 1996, 1997]. Since then, numerous works on the fractional calculus of variations, fractional optimal control and its applications have been written—see, e.g., [Agrawal, 2002, Agrawal and Baleanu, 2007, Almeida and Torres, 2009a, 2010, Atanacković et al., 2008, Baleanu, 2008, El-Nabulsi and Torres, 2007, Frederico and Torres, 2007, 2008, Klimek, 2002, Malinowska and Torres, 2010] and references therein. For the study of fractional isoperimetric problems see [Almeida et al., 2009].

In the pioneering paper [Agrawal, 2002], and others that followed, the fractional necessary optimality conditions are proved under the hypothesis that admissible functions y have continuous left and right fractional derivatives on the closed interval $[a, b]$. By considering that the admissible functions y have continuous left fractional derivatives on the whole interval, then necessarily $y(a) = 0$; by considering that the admissible functions y have continuous right fractional derivatives, then necessarily $y(b) = 0$. This fact has been independently remarked, in different contexts, at least in [Almeida et al., 2009, Almeida and Torres, 2010, Atanacković et al., 2008, Jelicic and Petrovacki, 2009]. In our work we want to be able to consider arbitrarily given boundary conditions $y(a) = y_a$ and $y(b) = y_b$ (and isoperimetric constraints). For that we consider variational functionals with integrands involving not only a fractional derivative of order $\alpha \in (0, 1)$ of the unknown function y, but also the classical derivative y'. More precisely, we consider dependence of the integrands on the independent variable t, unknown function y, and $y' + k\,{}_aD_t^\alpha y$ with k a real parameter. As a consequence, one gets a proper extension of the classical calculus of variations, in the sense that the classical theory is recovered with the particular situation $k = 0$. We remark that this is not the case with all the previous literature on the fractional variational calculus, where the classical theory is not included as a particular case and only as a limit, when $\alpha \to 1$.

The text is organized as follows. In Section 2 we briefly recall the necessary definitions and properties of the fractional calculus in the sense of Riemann-Liouville. Our results are stated, proved, and illustrated through an example, in Section 3. We end with Section 4 of conclusion.

⋆ This work is part of the first author's PhD project, which is carried out at the University of Aveiro under the Doctoral Programme *Mathematics and Applications* of Universities of Aveiro and Minho.

2. PRELIMINARIES

In this section some basic definitions and properties of fractional calculus are presented. For more on the subject we refer the reader to the books [Kilbas et al., 2006, Miller and Ross, 1993, Podlubny, 1999].

Definition 1. (Left and right Riemann-Liouville derivatives). Let f be a function defined on $[a,b]$. The operator ${}_aD_t^\alpha$,

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \,,$$

is called the left Riemann-Liouville fractional derivative of order α, and the operator ${}_tD_b^\alpha$,

$${}_tD_b^\alpha f(t) = \frac{-1}{\Gamma(n-\alpha)} D^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau \,,$$

is called the right Riemann-Liouville fractional derivative of order α, where $\alpha \in \mathbb{R}^+$ is the order of the derivatives and the integer number n is such that $n-1 \leq \alpha < n$.

Definition 2. (Mittag-Leffler function). Let $\alpha, \beta > 0$. The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \,.$$

Theorem 3. (Integration by parts). If f, g and the fractional derivatives ${}_aD_t^\alpha g$ and ${}_tD_b^\alpha f$ are continuous at every point $t \in [a,b]$, then

$$\int_a^b f(t) {}_aD_t^\alpha g(t) dt = \int_a^b g(t) {}_tD_b^\alpha f(t) dt \tag{1}$$

for any $0 < \alpha < 1$.

Remark 4. If $f(a) \neq 0$, then ${}_aD_t^\alpha f(t)|_{t=a} = \infty$. Similarly, if $f(b) \neq 0$, then ${}_tD_b^\alpha f(t)|_{t=b} = \infty$. Thus, if f possesses continuous left and right Riemann-Liouville fractional derivatives at every point $t \in [a,b]$, then $f(a) = f(b) = 0$. This explains why the usual term $f(t)g(t)|_a^b$ does not appear on the right-hand side of (1).

3. MAIN RESULTS

Following [Jelicic and Petrovacki, 2009], we prove optimality conditions of Euler-Lagrange type for variational problems containing classical and fractional derivatives simultaneously. In Section 3.1 the fundamental variational problem is considered, while in Section 3.2 we study the isoperimetric problem. Our results cover fractional variational problems subject to arbitrarily given boundary conditions. This is in contrast with [Agrawal, 2002, 2008, Agrawal and Baleanu, 2007, Baleanu et al., 2009], where the necessary optimality conditions are valid for appropriate zero valued boundary conditions (cf. Remark 4). For a discussion on this matter see [Almeida and Torres, 2010, Atanacković et al., 2008, Jelicic and Petrovacki, 2009].

3.1 The Euler-Lagrange equation

Let $0 < \alpha < 1$. Consider the following problem: find a function $y \in C^1[a,b]$ for which the functional

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t)\right) dt \tag{2}$$

subject to given boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{3}$$

has an extremum. We assume that k is a fixed real number, $F \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$, and $\partial_3 F$ (the partial derivative of $F(\cdot,\cdot,\cdot)$ with respect to its third argument) has a continuous right Riemann-Liouville fractional derivative of order α.

Definition 5. A function $y \in C^1[a,b]$ that satisfies the given boundary conditions (3) is said to be *admissible* for problem (2)-(3).

For simplicity of notation we introduce the operator $[y]_k^\alpha$ defined by

$$[y]_k^\alpha(t) = (t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t)) \,.$$

With this notation we can write (2) simply as

$$\mathcal{J}(y) = \int_a^b F[y]_k^\alpha(t) dt \,.$$

Theorem 6. (The fractional Euler-Lagrange equation). If y is an extremizer (minimizer or maximizer) of problem (2)-(3), then y satisfies the Euler-Lagrange equation

$$\partial_2 F[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 F[y]_k^\alpha(t) + k\, {}_tD_b^\alpha \partial_3 F[y]_k^\alpha(t) = 0 \tag{4}$$

for all $t \in [a,b]$.

Proof. Suppose that y is a solution of (2)-(3). Note that admissible functions $\hat{y}$ can be written in the form $\hat{y}(t) = y(t) + \epsilon\eta(t)$, where $\eta \in C^1[a,b]$, $\eta(a) = \eta(b) = 0$, and $\epsilon \in \mathbb{R}$. Let $J(\epsilon) = \int_a^b F(t, y(t) + \epsilon\eta(t), \frac{d}{dt}(y(t) + \epsilon\eta(t)) + k_a D_t^\alpha (y(t) + \epsilon\eta(t)))dt$. Since ${}_aD_t^\alpha$ is a linear operator, we know that

$${}_aD_t^\alpha (y(t) + \epsilon\eta(t)) = {}_a D_t^\alpha y(t) + \epsilon {}_aD_t^\alpha \eta(t).$$

On the other hand,

$$\begin{aligned} \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} &= \left.\int_a^b \frac{d}{d\epsilon} F[\hat{y}]_k^\alpha(t) dt\right|_{\epsilon=0} \\ &= \int_a^b \Big(\partial_2 F[y]_k^\alpha(t) \cdot \eta(t) + \partial_3 F[y]_k^\alpha(t) \frac{d\eta(t)}{dt} \\ &\quad + k \partial_3 F[y]_k^\alpha(t) {}_aD_t^\alpha \eta(t)\Big) dt. \end{aligned} \tag{5}$$

Using integration by parts we get

$$\int_a^b \partial_3 F \frac{d\eta}{dt} dt = \partial_3 F \eta|_a^b - \int_a^b (\eta \frac{d}{dt}\partial_3 F) dt \tag{6}$$

and

$$k \int_a^b \partial_3 F_a D_t^\alpha \eta dt = \int_a^b \eta_t D_b \partial_3 F dt. \tag{7}$$

Substituting (6) and (7) into (5), and having in mind that $\eta(a) = \eta(b) = 0$, it follows that

$$\begin{aligned} \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} &= \int_a^b \eta(t) \Big(\partial_2 F[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 F[y]_k^\alpha(t) \\ &\quad + k\, {}_tD_b^\alpha \partial_3 F[y]_k^\alpha(t)\Big) dt. \end{aligned}$$

A necessary optimality condition is given by $\frac{dJ}{d\epsilon}\big|_{\epsilon=0} = 0$. Hence,

$$\begin{aligned} \int_a^b \eta(t) \Big(\partial_2 F[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 F[y]_k^\alpha(t) \\ + k\, {}_tD_b^\alpha \partial_3 F[y]_k^\alpha(t)\Big) dt = 0. \end{aligned} \tag{8}$$

We obtain (4) applying the fundamental lemma of the calculus of variations to (8).

Remark 7. Note that for $k = 0$ our necessary optimality condition (4) reduces to the classical Euler-Lagrange equation [Gelfand and Fomin, 1963, van Brunt, 2004].

3.2 The fractional isoperimetric problem

As before, let $0 < \alpha < 1$. We now consider the problem of extremizing a functional

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t)\right) dt \tag{9}$$

in the class $y \in C^1[a,b]$ when subject to given boundary conditions

$$y(a) = y_a\,, \quad y(b) = y_b, \tag{10}$$

and an isoperimetric constraint

$$\mathcal{I}(y) = \int_a^b G(t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t))dt = \xi\,. \tag{11}$$

We assume that k and ξ are fixed real numbers, $F, G \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$, and $\partial_3 F$ and $\partial_3 G$ have continuous right Riemann-Liouville fractional derivatives of order α.

Definition 8. A function $y \in C^1[a,b]$ that satisfies the given boundary conditions (10) and isoperimetric constraint (11) is said to be *admissible* for problem (9)-(11).

Definition 9. An admissible function y is an *extremal* for $\mathcal{I}$ if it satisfies the fractional Euler-Lagrange equation

$$\partial_2 G[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 G[y]_k^\alpha(t) + k\, {}_tD_b^\alpha \partial_3 G[y]_k^\alpha(t) = 0$$

for all $t \in [a,b]$.

The next theorem gives a necessary optimality condition for the fractional isoperimetric problem (9)-(11).

Theorem 10. Let y be an extremizer to the functional (9) subject to the boundary conditions (10) and the isoperimetric constraint (11). If y is not an extremal for $\mathcal{I}$, then there exists a constant λ such that

$$\partial_2 H[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 H[y]_k^\alpha(t) + k\, {}_tD_b^\alpha \partial_3 H[y]_k^\alpha(t) = 0 \tag{12}$$

for all $t \in [a,b]$, where $H(t,y,v) = F(t,y,v) - \lambda G(t,y,v)$.

Proof. We introduce the two parameter family

$$\hat{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \tag{13}$$

in which η_1 and η_2 are such that $\eta_1, \eta_2 \in C^1[a,b]$ and they have continuous left and right fractional derivatives. We also require that

$$\eta_1(a) = \eta_1(b) = 0 = \eta_2(a) = \eta_2(b).$$

First we need to show that in the family (13) there are curves such that $\hat{y}$ satisfies (11). Substituting y by $\hat{y}$ in (11), $\mathcal{I}(\hat{y})$ becomes a function of two parameters ϵ_1, ϵ_2. Let

$$\hat{I}(\epsilon_1, \epsilon_2) = \int_a^b G(t, \hat{y}, \hat{y}' + k_a D_t^\alpha \hat{y})dt - \xi.$$

Then, $\hat{I}(0,0) = 0$ and

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} = \int_a^b \eta_2 \left(\partial_2 G - \frac{d}{dt}\partial_3 G + k_t D_b^\alpha \partial_3 G\right) dt.$$

Since y is not an extremal for $\mathcal{I}$, by the fundamental lemma of the calculus of variations there is a function η_2 such that

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} \neq 0.$$

By the implicit function theorem, there exists a function $\epsilon_2(\cdot)$ defined in a neighborhood of $(0,0)$ such that

$$\hat{I}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0.$$

Let $\hat{J}(\epsilon_1, \epsilon_2) = \mathcal{J}(\hat{y})$. Then, by the Lagrange multiplier rule, there exists a real λ such that

$$\nabla(\hat{J}(0,0) - \lambda \hat{I}(0,0)) = \mathbf{0}.$$

Because

$$\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \eta_1 \left(\partial_2 F - \frac{d}{dt}\partial_3 F + k_t D_b^\alpha \partial_3 F\right) dt$$

and

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \eta_1 \left(\partial_2 G - \frac{d}{dt}\partial_3 G + k_t D_b^\alpha \partial_3 G\right) dt,$$

one has

$$\int_a^b \eta_1 \left[\left(\partial_2 F - \frac{d}{dt}\partial_3 F + k_t D_b^\alpha \partial_3 F\right) - \lambda \left(\partial_2 G - \frac{d}{dt}\partial_3 G + k_t D_b^\alpha \partial_3 G\right)\right] dt = 0.$$

Since η_1 is an arbitrary function, (12) follows from the fundamental lemma of the calculus of variations.

3.3 An example

Let $\alpha \in (0,1)$ and $k, \xi \in \mathbb{R}$. Consider the following fractional isoperimetric problem:

$$\begin{gathered} \mathcal{J}(y) = \int_0^1 \left(y' + k\, {}_0D_t^\alpha y\right)^2 dt \longrightarrow \min \\ \mathcal{I}(y) = \int_0^1 \left(y' + k\, {}_0D_t^\alpha y\right) dt = \xi \\ y(0) = 0\,,\ y(1) = \int_0^1 E_{1-\alpha,1}\left(-k\,(1-\tau)^{1-\alpha}\right) \xi d\tau. \end{gathered} \tag{14}$$

In this case the augmented Lagrangian H of Theorem 10 is given by $H(t,y,v) = v^2 - \lambda v$. One can easily check that

$$y(t) = \int_0^t E_{1-\alpha,1}\left(-k\,(t-\tau)^{1-\alpha}\right) \xi d\tau \tag{15}$$

- is not an extremal for $\mathcal{I}$;
- satisfies $y' + k\, {}_0D_t^\alpha y = \xi$ (see, e.g., [Kilbas et al., 2006, p. 297, Theorem 5.5]).

Moreover, (15) satisfies (12) for $\lambda = 2\xi$, i.e.,

$$-\frac{d}{dt}\left(2\left(y' + k\, {}_0D_t^\alpha y\right) - 2\xi\right) + k\, {}_tD_1^\alpha \left(2\left(y' + k\, {}_0D_t^\alpha y\right) - 2\xi\right) = 0.$$

We conclude that (15) is the extremal for problem (14).

Example 11. Choose $k = 0$. In this case the isoperimetric constraint is trivially satisfied, (14) is reduced to the classical problem of the calculus of variations

$$\begin{gathered} \mathcal{J}(y) = \int_0^1 (y'(t))^2 dt \longrightarrow \min \\ y(0) = 0\,, \quad y(1) = \xi, \end{gathered} \tag{16}$$

and our general extremal (15) simplifies to the well-known minimizer $y(t) = \xi t$ of (16).

Example 12. When $\alpha \to 1$ the isoperimetric constraint is redundant with the boundary conditions, and the fractional problem (14) simplifies to the classical variational problem

$$\begin{gathered} \mathcal{J}(y) = (k+1)^2 \int_0^1 y'(t)^2 dt \longrightarrow \min \\ y(0) = 0\,, \quad y(1) = \frac{\xi}{k+1}. \end{gathered} \tag{17}$$

Our fractional extremal (15) gives $y(t) = \frac{\xi}{k+1} t$, which is exactly the minimizer of (17).

Example 13. Choose $k = \xi = 1$. When $\alpha \to 0$ one gets from (14) the classical isoperimetric problem

$$\begin{gathered} \mathcal{J}(y) = \int_0^1 \left(y'(t) + y(t)\right)^2 dt \longrightarrow \min \\ \mathcal{I}(y) = \int_0^1 y(t) dt = \frac{1}{e} \\ y(0) = 0\,, \quad y(1) = 1 - \frac{1}{\mathrm{e}}. \end{gathered} \tag{18}$$

Our extremal (15) is then reduced to the classical extremal $y(t) = 1 - \mathrm{e}^{-t}$ of (18).

Example 14. Choose $k = 1$ and $\alpha = \frac{1}{2}$. Then (14) gives the following fractional isoperimetric problem:

$$\begin{gathered} \mathcal{J}(y) = \int_0^1 \left(y' + {}_0D_t^{\frac{1}{2}} y\right)^2 dt \longrightarrow \min \\ \mathcal{I}(y) = \int_0^1 \left(y' + {}_0D_t^{\frac{1}{2}} y\right) dt = \xi \\ y(0) = 0\,, \quad y(1) = -\xi \left(1 - \operatorname{erfc}(1) + \frac{2}{\sqrt{\pi}}\right), \end{gathered} \tag{19}$$

where erfc is the complementary error function. The extremal (15) for the particular fractional problem (19) is

$$y(t) = -\xi \left(1 - \mathrm{e}^t \operatorname{erfc}(\sqrt{t}) + \frac{2\sqrt{t}}{\sqrt{\pi}}\right).$$

4. CONCLUSION

Fractional variational calculus provides a very useful framework to deal with nonlocal dynamics in Mechanics and Physics [Baleanu and Trujillo, 2010]. Motivated by the results and insights of [Almeida et al., 2009, Almeida and Torres, 2009a, Jelicic and Petrovacki, 2009], in this paper we generalize previous fractional Euler-Lagrange equations by proving optimality conditions for fractional problems of the calculus of variations where the highest derivative in the Lagrangian is of integer order. This approach avoids difficulties with the given boundary conditions when in presence of Riemann–Liouville derivatives [Jelicic and Petrovacki, 2009].

We focus our attention to problems subject to integral constraints (fractional isoperimetric problems), which have recently found a broad class of important applications [Almeida and Torres, 2009b, Blåsjö, 2005, Curtis, 2004]. For $k = 0$ our results are reduced to the classical ones [van Brunt, 2004]. This is in contrast with the standard approach to fractional variational calculus, where the classical (integer-order) case is obtained only in the limit.

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REFERENCES

N. H. Abel. *Euvres completes de Niels Henrik Abel.* Christiana: Imprimerie de Grondahl and Son; New York and London: Johnson Reprint Corporation. VIII, 621 pp., 1965.

O. P. Agrawal. Formulation of Euler-Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.*, 272 (1): 368–379, 2002.

O. P. Agrawal. A general finite element formulation for fractional variational problems. *J. Math. Anal. Appl.*, 337 (1): 1–12, 2008.

O. P. Agrawal and D. Baleanu. A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems. *J. Vib. Control*, 13 (9-10): 1269–1281, 2007.

R. Almeida and D. F. M. Torres. Calculus of variations with fractional derivatives and fractional integrals. *Appl. Math. Lett.*, 22 (12): 1816–1820, 2009a.

R. Almeida and D. F. M. Torres. Hölderian variational problems subject to integral constraints. *J. Math. Anal. Appl.*, 359 (2): 674–681, 2009b.

R. Almeida and D. F. M. Torres. Leitmann's direct method for fractional optimization problems. *Appl. Math. Comput.*, in press, 2010. DOI: 10.1016/j.amc.2010.03.085

R. Almeida, R. A. C. Ferreira, and D. F. M. Torres. Isoperimetric problems of the calculus of variations with fractional derivatives. Submitted, 2009.

T. M. Atanacković, S. Konjik, and S. Pilipović. Variational problems with fractional derivatives: Euler-Lagrange equations. *J. Phys. A*, 41 (9): 095201, 12, 2008.

D. Baleanu. New applications of fractional variational principles. *Rep. Math. Phys.*, 61 (2): 199–206, 2008.

D. Baleanu and J. I. Trujillo. A new method of finding the fractional Euler-Lagrange and Hamilton equations within Caputo fractional derivatives. *Communications in Nonlinear Science and Numerical Simulation*, 15 (5): 1111 – 1115, 2010.

D. Baleanu, O. Defterli, and O. P. Agrawal. A central difference numerical scheme for fractional optimal control problems. *J. Vib. Control*, 15 (4): 583–597, 2009.

V. Blåsjö. The isoperimetric problem. *Amer. Math. Monthly*, 112 (6): 526–566, 2005.

J. P. Curtis. Complementary extremum principles for isoperimetric optimization problems. *Optim. Eng.*, 5 (4): 417–430, 2004.

R. A. El-Nabulsi and D. F. M. Torres. Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β). *Math. Methods Appl. Sci.*, 30 (15): 1931–1939, 2007.

G. S. F. Frederico and D. F. M. Torres. A formulation of Noether's theorem for fractional problems of the calculus of variations. *J. Math. Anal. Appl.*, 334 (2): 834–846, 2007.

G. S. F. Frederico and D. F. M. Torres. Fractional conservation laws in optimal control theory. *Nonlinear Dynam.*, 53 (3): 215–222, 2008.

I. M. Gelfand and S. V. Fomin. *Calculus of variations*. Revised English edition translated and edited by Richard A. Silverman. Prentice-Hall Inc., Englewood Cliffs, N.J., 1963.

Z. D. Jelicic and N. Petrovacki. Optimality conditions and a solution scheme for fractional optimal control problems. *Struct. Multidiscip. Optim.*, 38 (6): 571–581, 2009.

A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.

M. Klimek. Lagrangean and Hamiltonian fractional sequential mechanics. *Czechoslovak J. Phys.*, 52 (11): 1247–1253, 2002.

A. B. Malinowska and D. F. M. Torres. Generalized natural boundary conditions for fractional variational problems in terms of the caputo derivative. *Comput. Math. Appl.*, 59 (9): 3110–3116, 2010.

K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1993.

I. Podlubny. *Fractional differential equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press Inc., San Diego, CA, 1999.

F. Riewe. Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E (3)*, 53 (2): 1890–1899, 1996.

F. Riewe. Mechanics with fractional derivatives. *Phys. Rev. E (3)*, 55 (3, part B): 3581–3592, 1997.

B. van Brunt. *The calculus of variations*. Universitext. Springer-Verlag, New York, 2004.