Estimates for the equivalence constant of the torsional rigidity coefficient and the Euclidean moment of inertia

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In the **Saint-Venant** classical work, torsional rigidity coefficient is defined by the following expression:

$$P(\Omega) = 2 \iint\limits_{\Omega} v \, dxdy,$$

here $v = v(x, y)$ is the solution of the Dirichlet problem for Poisson equation $\Delta v = -2$ with zero boundary condition $v|_{\partial\Omega} = 0$.

Farit Avkhadiev introduced (1998) a functional, called Euclidean moment of inertia (moment of inertia about boundary). It is defined as follows

$$I(\partial\Omega) = \iint_{\Omega} dist(x,y)^2 \, dxdy,$$

here $dist(x,y)$ is the distance from point $(x,y) \in \Omega$ to domain boundary $\partial\Omega$.

The equivalence between the torsional rigidity coefficient and Euclidean moment of inertia means that there are constants μ, λ such that

$$\mu I(\partial\Omega) \leq P(\Omega) \leq \lambda I(\partial\Omega)$$

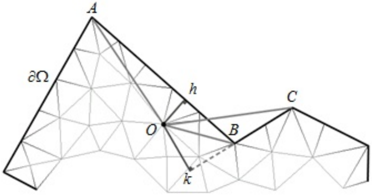
$$\mu = \inf_{\Omega} P(\Omega)/I(\partial\Omega), \quad \lambda = \sup_{\Omega} P(\Omega)/I(\partial\Omega)$$

for any simply connected domain.

- $\lambda \leq 64$ for simply connected domains. [Avkhadiev (1998)]
- $\lambda \leq 4$ for convex domains. [Avkhadiev (1998)]
- $\lambda > 4$ for non-convex domains. [Kovalev (2002)]
- $\lambda \geq 4.08$ for non-convex domains. [Giniyatova, Salakhudinov (2007)]

We use similar approaches to calculate $P(\Omega)$ and $I(\partial\Omega)$

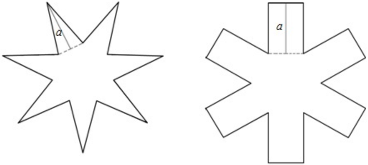
- Stress function: FEM + CG
- Distance: our own approach



$$dist(x, y) = \min_i \delta_i$$

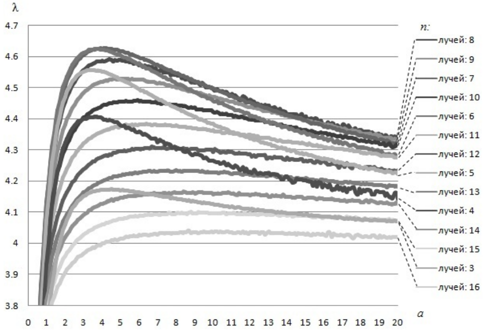
For the i-triangle with sides α, β, γ ($\gamma \in \partial\Omega$)

$$\delta_i^2 = \begin{cases} \alpha^2 - \frac{\left(\gamma^2+\alpha^2-\beta^2\right)^2}{4\gamma^2}, & \alpha^2+\gamma^2 > \beta^2 \; and \; \beta^2+\gamma^2 > \alpha^2, \\ min\left(\alpha^2, \beta^2\right), & \alpha^2+\gamma^2 \le \beta^2 \; or \; \beta^2+\gamma^2 \le \alpha^2. \end{cases}$$

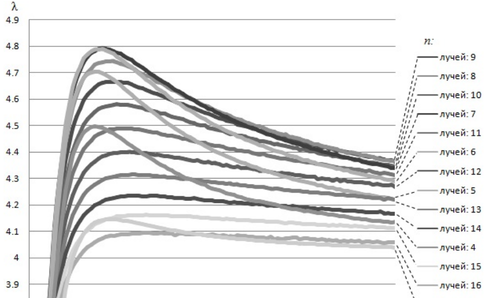


- The class ST of starshaped domains with two parameters n and a. n is the number of rays, a is the relation between the ray height and the ray base.
- The class SN of the so called "snowflaked"domains with the similar parameters.

$ST(n, a)$ is a starshaped domain with parameters n and a.



$SN(n, a)$ is a snowflaked domain with parameters n and a.



$\lambda(K) = \sup\limits_{\Omega \in K} P(\Omega)/I(\partial\Omega)$ the sharp equivalence constant for some class K.

As the results of the numerical analysis we can conclude the following:

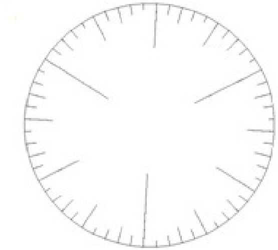
1. In the class ST there is the domain Ω_1 ($n = 7$, $a \approx 3.9$) such that $P(\Omega_1) \approx 4.64 I(\partial\Omega_1)$ and therefore the following estimate holds

$$\lambda \geq \lambda(ST) \geq 4.64.$$

2. In the class SN there is the domain Ω_2 ($n = 7$, $a \approx 2$) such that $P(\Omega_2) \approx 4.8 I(\partial\Omega_2)$ and therefore the following estimate holds

$$\lambda \geq \lambda(SN) \geq 4.8.$$

Then we consider the class of disks with radial symmetric cuts.
We find that it is better to take disks with recursively decreasing cuts.
In such way we construct the domain Ω_3.



For this domain we have $P(\Omega_3) \approx 6I(\partial\Omega_3)$.

For non-convex simply connected domains the following estimates hold

$$6 \leq \lambda \leq 64$$