The isoperimetric problem in surfaces of revolution

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The isoperimetric problem in a surface

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- Bounded by closed embedded curves with constant geodesic curvature

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- Existence

- Bounded by closed embedded curves with constant geodesic curvature

- Classified for some surfaces

The isoperimetric problem in a surface

- Plane: Disks

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- Sphere: Geodesic disks

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- Paraboloid: Geodesic disks centered at the origin

I. Benjamini and J. Cao, 1996

The isoperimetric problem in a surface

- Plane: Disks
- Sphere: Geodesic disks
- Right cylinder: Geodesic disks and horizontal strips
- Paraboloid: Geodesic disks centered at the origin
- Planes and spheres with monotonic Gauss curvature

F. Morgan, M. Hutchings and H. Howards, 2000; M. Ritoré, 2001

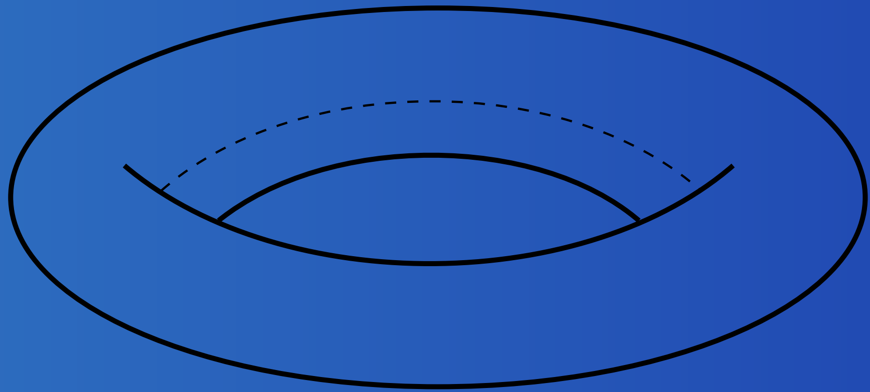
Our work

We study the isoperimetric problem in:

- symmetric tori of revolution with decreasing Gauss curvature
- symmetric annuli of revolution with increasing Gauss curvature

Our work

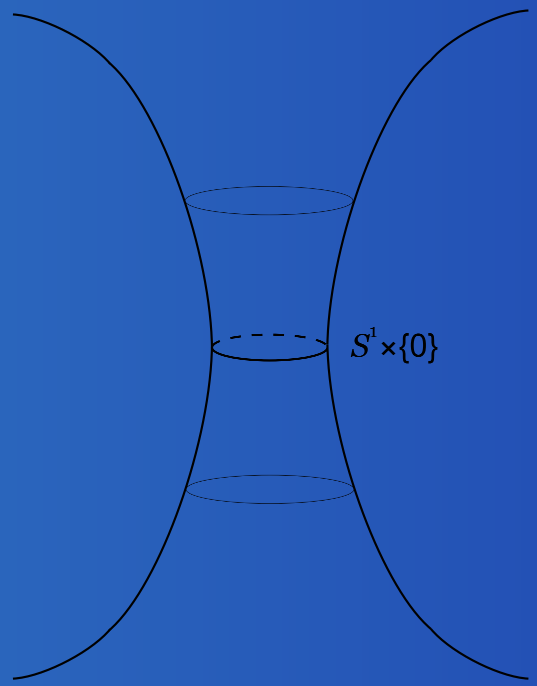
- Symmetric tori of revolution with decreasing Gauss curvature:



Standard torus of revolution

Our work

- Symmetric annuli of revolution with increasing Gauss curvature:



One-sheeted hyperboloid and catenoid

Our approach

- Isoperimetric regions are bounded by constant geodesic curvature curves

- Isoperimetric regions are stable regions

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$\rightsquigarrow$ isoperimetric candidates

Constant geodesic curvature curves

$M \subset \mathbb{R}^3$ surface of revolution

We will see M as a warped product $\mathbb{S}^1 \times I$ with metric

$$ds^2 = f(t)^2\, d\theta^2 + dt^2,$$

where $I \subset \mathbb{R}$ is a real interval, and $f : I \to \mathbb{R}^+$ is a C^1 real function

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$\rightsquigarrow$ Classification of the curves in M with constant geodesic curvature

Constant geodesic curvature curves

- Circles of revolution: $\mathbb{S}^1 \times \{t\}$, $t \in I$

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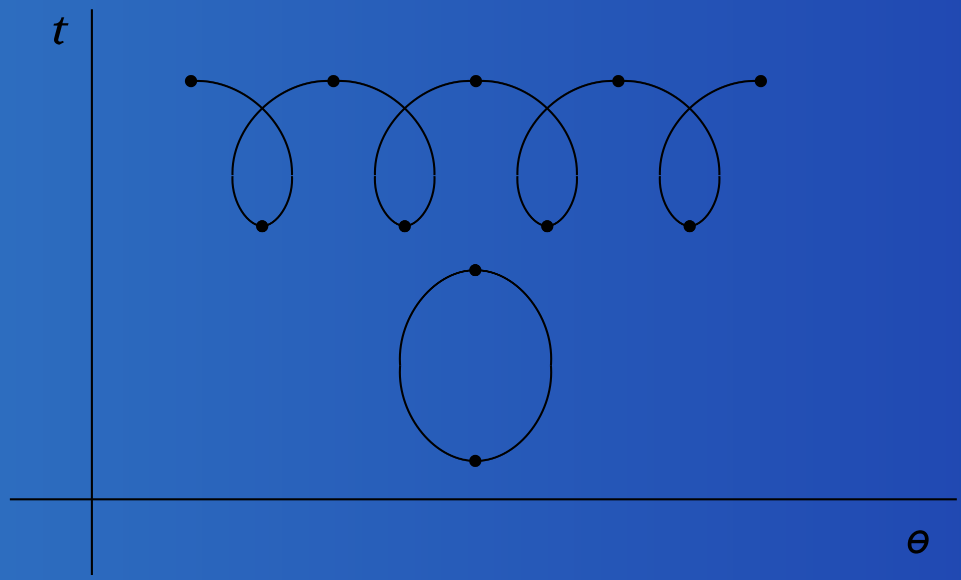
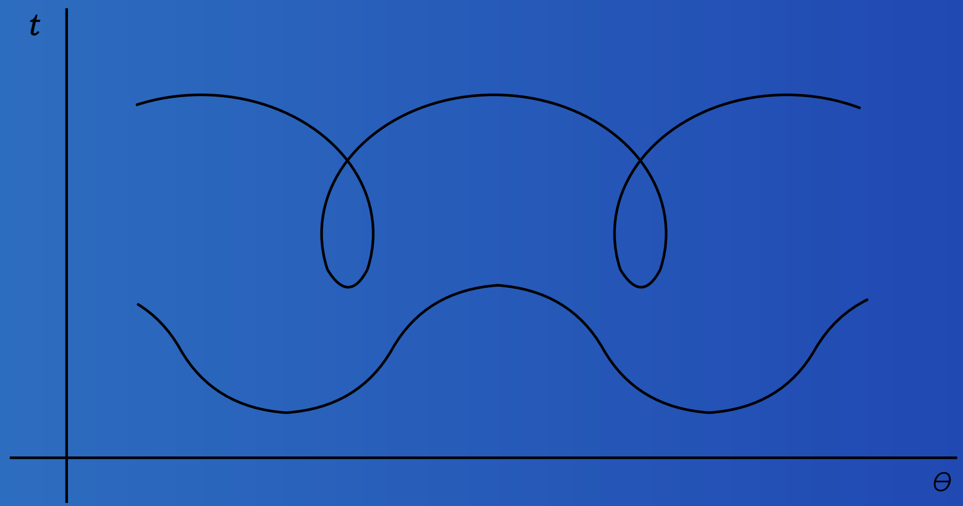


Constant geodesic curvature curves

- Circles of revolution: $\mathbb{S}^1 \times \{t\},\ t \in I$
- Nodoids (bounding disks when closed)
- Unduloids (in general, not closed and embedded)



Annuli with increasing Gauss curvature

In symmetric annuli of revolution with increasing Gauss curvature:

- Existence of isoperimetric regions is not guaranteed (catenoids)

Annuli with increasing Gauss curvature

In symmetric annuli of revolution with increasing Gauss curvature:

- Closed embedded curves with constant geodesic curvature:

  $$\left\{ \begin{array}{l} \text{- circles of revolution} \\ \text{- nodoids} \\ \text{- unduloids} \end{array} \right.$$

Annuli with increasing Gauss curvature

The stable regions are

i) disks bounded by nodoids with constant Gauss curvature

Annuli with increasing Gauss curvature

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i) disks bounded by nodoids with constant Gauss curvature

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

Annuli with increasing Gauss curvature

The stable regions are

i) disks bounded by nodoids with constant Gauss curvature

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

Annuli with increasing Gauss curvature

The stable regions are

i) disks bounded by nodoids with constant Gauss curvature

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

iv) unions of a disk and a symmetric annulus

Annuli with increasing Gauss curvature

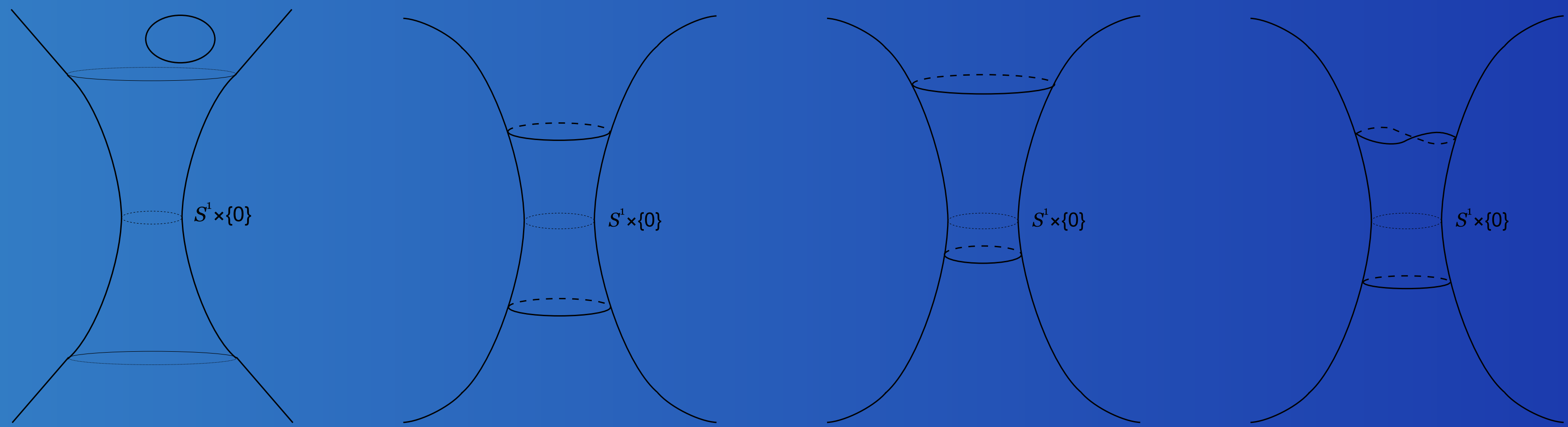
Finally, the isoperimetric regions are

i) disks with constant Gauss curvature (equal to its maximum)

ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

iii) annuli bounded by an unduloid and a circle of revolution

Annuli with increasing Gauss curvature



Isoperimetric regions in symmetric annuli of revolution

with increasing Gauss curvature

Tori with decreasing Gauss curvature

$\rightsquigarrow$ We can see the torus as a warped product (finite annulus):

Tori with decreasing Gauss curvature

$\rightsquigarrow$ We can see the torus as a warped product (finite annulus):

$$M = \mathbb{S}^1 \times [-t_0, t_0], \quad ds^2 = f(t)^2\, d\theta^2 + dt^2$$

By identifying $\mathbb{S}^1 \times \{-t_0\}$ and $\mathbb{S}^1 \times \{t_0\}$

$\longrightarrow$ Torus of revolution

Tori with decreasing Gauss curvature

In symmetric tori of revolution with decreasing Gauss curvature:

- Existence of isoperimetric regions is guaranteed by compactness

Tori with decreasing Gauss curvature

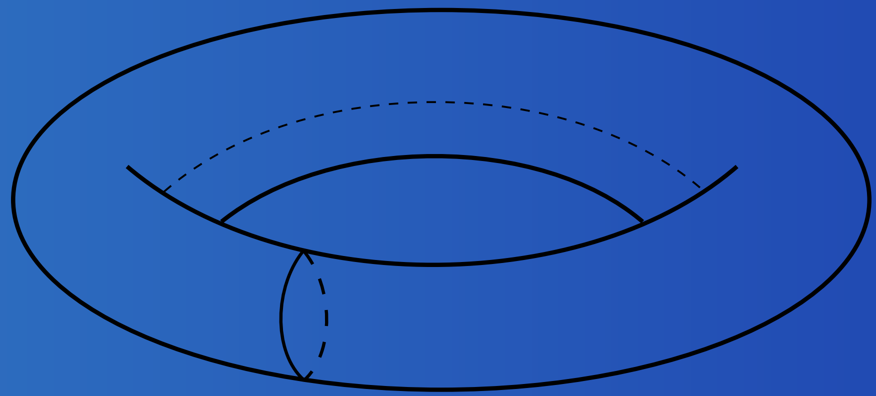
In symmetric tori of revolution with decreasing Gauss curvature:

- Closed embedded curves with constant geodesic curvature:
  - circles of revolution
  - nodoids
  - unduloids
  - vertical geodesics
  - helix type curves

Tori with decreasing Gauss curvature

- Vertical geodesics:

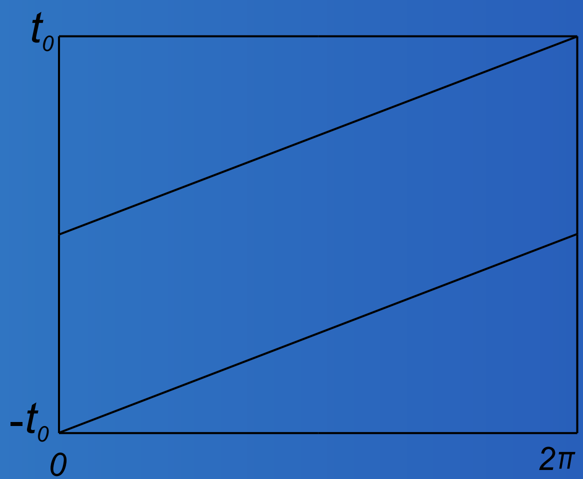
  Generating curves of the torus of revolution

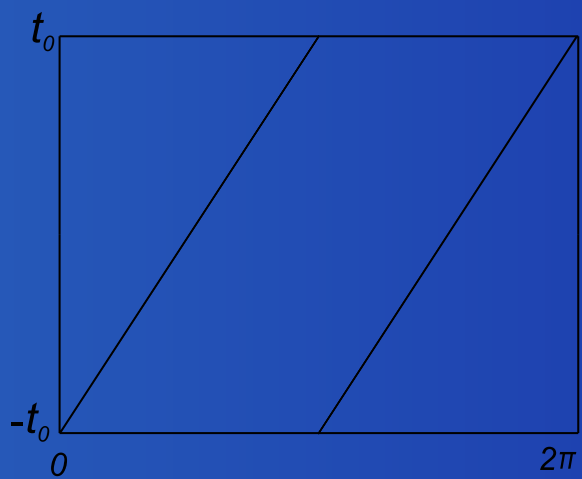


Tori with decreasing Gauss curvature

- Helix type curves:

  Geodesics in the torus (not closed in general)





Two different helix type curves in $[0, 2\pi] \times [-t_0, t_0]$

Tori with decreasing Gauss curvature

The stable regions are

i) disks bounded by nodoids
(symmetric or with constant Gauss curvature)

Tori with decreasing Gauss curvature

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ii) annuli bounded by two circles of revolution (symmetric or non-symmetric)

Tori with decreasing Gauss curvature

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iv) unions of a disk and a symmetric annulus

Tori with decreasing Gauss curvature

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iv) unions of a disk and a symmetric annulus

v) unions of vertical annuli (bounded by vertical geodesics)

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iii) annuli bounded by an unduloid and a circle of revolution

iv) unions of a disk and a symmetric annulus

v) unions of vertical annuli

vi) unions of annuli bounded by helix type curves

Tori with decreasing Gauss curvature

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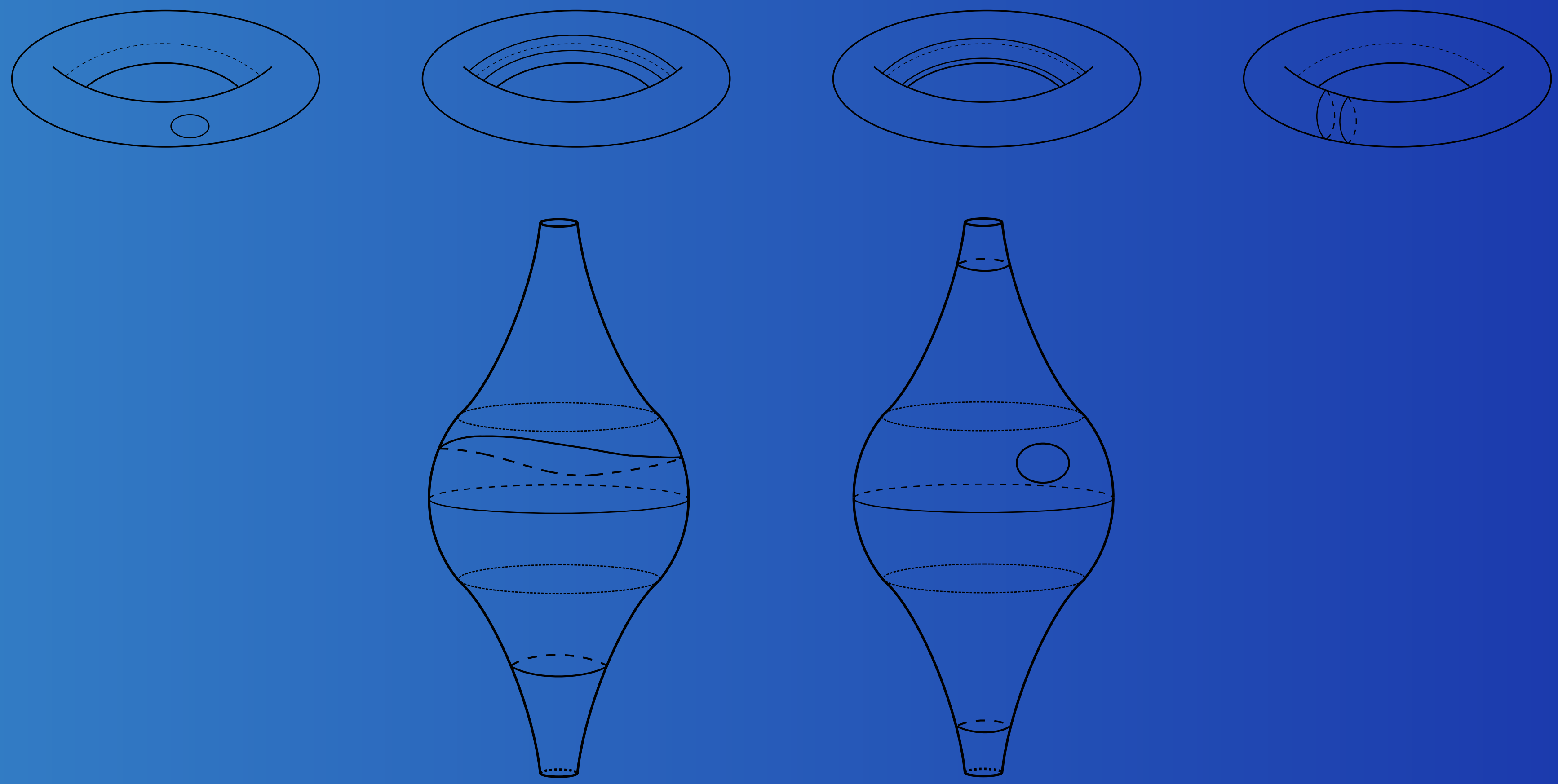
Tori with decreasing Gauss curvature



Isoperimetric regions in symmetric tori of revolution with decreasing Gauss curvature

Main consequences

- Unduloids may appear in the isoperimetric boundaries

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- The Gauss curvature of the surfaces may be piecewise continuous