Best Sobolev constants with critical growth on spheres

joint work with
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Motivation

$$D \subset \mathbb{R}^N : \text{ bounded domain },$$

$$1 < p < N, \quad p^* = \frac{Np}{N-p} : \text{ critical Sobolev exponent },$$

$$\mathcal{K} := \left\{ v \in W_0^{1,p}(D) : \int_D |v|^{p^*} \, dx = 1 \right\}.$$

Consider the critical Sobolev constant

$$\boxed{S_p(D) := \inf_{\mathcal{K}} \int_D |\nabla v|^p \, dx}.$$

Known result

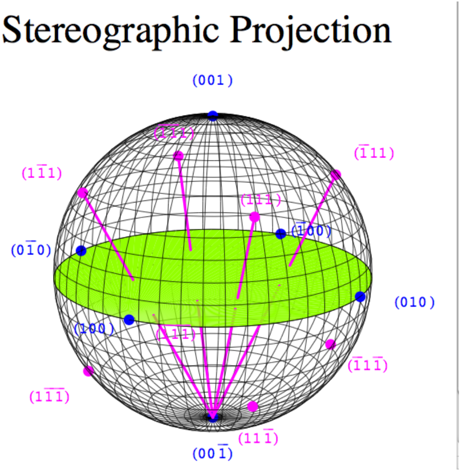
- $S_p(D)$ is never attained.
- $S_p(D) = S_p^*$ for all domains.
- All minimizing sequences concentrate.
  Minimizing sequence

$$u_\epsilon = (\epsilon + r^{p/(p-1)})^{\frac{N-p}{p}} \underbrace{\psi}_{\text{cutoff function}}$$

What happens if we look at the same Sobolev constant in $\mathbb{S}^N$?

Analytic formulation

$$\mathbb{S}^N = \{x \in \mathbb{R}^{N+1} : |x| = 1\}$$



Let $P \in \mathbb{S}^N$ and denote by $\mathcal{S}P \in \mathbb{R}^N$ its stereographic projection.

Distance between P and P' on $\mathbb{S}^N$:

$$d(P,P') = \frac{2|\mathcal{S}P - \mathcal{S}P'|}{\sqrt{(1+|\mathcal{S}P|^2)(1+|\mathcal{S}P'|^2)}}.$$

Model of $\mathbb{S}^N$:

$$(\mathbb{R}^N, ds = \underbrace{\frac{2}{1+|x|^2}}_{\rho(x)}|dx|).$$

$$|\nabla v|^2 \longrightarrow \rho^{-2}|\nabla v|^2,$$

$$\text{volume element } dx \longrightarrow \rho^N dx.$$

$\mathbb{R}^N$

$$\int_D |\nabla v|^p \, dx,$$

$$\int_D |v|^{p^*} \, dx,$$

$$S_p(D) = \inf_{W_0^{1,p}(D)} \frac{\int_D |\nabla v|^p \, dx}{\left(\int_D |v|^{p^*} \, dx\right)^{p/p^*}}.$$

$\mathbb{S}^N$

$$\int_D |\nabla v|^p \rho^{N-p} \, dx,$$

$$\int_D |v|^{p^*} \rho^N \, dx,$$

$$\tilde{S}_p(D) = \inf_{W_0^{1,p}(D)} \frac{\int_D |\nabla v|^p \rho^{N-p} \, dx}{\left(\int_D |v|^{p^*} \rho^N \, dx\right)^{p/p^*}}.$$

Euler-Lagrange equations

$\mathbb{R}^N$

$$\begin{aligned} u &> 0 \text{ in } D, \\ \triangle_p u + S_p(D) u^{p^*-1} &= 0, \\ u &= 0 \text{ on } \partial D. \end{aligned}$$

$\mathbb{S}^N$

$$\begin{aligned} u &> 0 \text{ in } D, \\ div(\rho^{N-p}|\nabla u|^{p-2}\nabla u) + \tilde{S}_p(D)\rho^N u^{p^*-1} &= 0, \\ u &= 0 \text{ on } \partial D. \end{aligned}$$

Every minimizer of $S_p(D)$ or $\tilde{S}_p(D)$ solves the corresponding Euler equation.

Properties of $\tilde{S}_p(D)$

- Monotonicity: If $D_1 \subset D_2$ then $\tilde{S}_p(D_1) > \tilde{S}_p(D_2)$
- Symmetry: If D^* is the geodesic ball with the same volume a D then $\tilde{S}_p(D) \geq \tilde{S}_p(D^*)$
- Universal upper bound

$$\tilde{S}_p(D) \leq \underbrace{S_p^*}_{\text{minimizer of } S_p(D)} .$$

- Concentration-compactness lemma If $\tilde{S}_p(D) < S_p^*$ then there exists a minimizer.

Geodesic balls

$$B_R := \{x \in \mathbb{R}^N : |x| < R\}$$

$\theta = 2\arctan(R)$: geodesic radius.

The minimizers are radially symmetric and solve

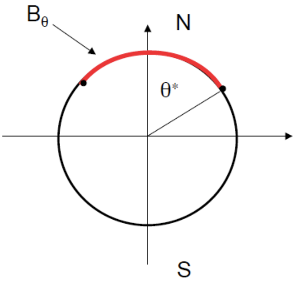
$$\begin{aligned}(\rho^{N-p}(r)r^{N-1}|u'|^{p-2}u')' + \rho^N rN-1u^{p^*-1} &= 0 \text{ in } (0,R),\\ u &> 0,\\ u'(0) = 0 \text{ and } u(R) &= 0. \qquad (1)\end{aligned}$$

Useful notation: $p = \frac{N+n}{n+1}$, $n > 0$.
Then $p^* = \frac{N(N+n)}{n(N-1)}$.

Nonexistence

Let $R_0(n)$ be the first zero of the hypergeometric function ${}_2F_1(-\frac{n}{2}, -n, 1-\frac{n}{2}, -r^2) \iff$ unique root of

$$2n \int_0^R \left(\frac{1}{t} + t\right)^{n-1} dt - \left(\frac{1}{R} + R\right)^n = 0.$$



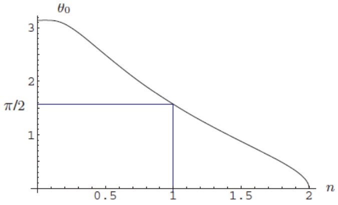


Fig. 1 Numerical computation of $\theta_0(n)$. The graph shows the dependence of $\theta_0 = 2\arctan R_0$ on $n \in (0,2)$.

Theorem

Assume that $n \in (0,2)$. *Then for* $0 < R < R_0(n)$ *no nontrivial solutions of* (1) *exists.*

$$R_0(n) \longrightarrow \begin{cases} \infty & \text{as } n \to 0, \\ 0 & \text{as } n \to 2. \end{cases}$$

Sketch of the proof

Test (1) by

$$a(r)u + b(r)u',$$

for suitably choosen $a(r), b(r)$.

$\Longrightarrow$ Pohozaev type identity.

Consequences

1. If $R < R_0(n)$ then
$$\tilde{S}_p(B_R) = S_p^*.$$
The minimum is not attained.
2. If $vol(D) \leq vol(B_R)$ for some $R < R_0(n)$ then
$$\tilde{S}_p(D) = S_p^*.$$
The minimum is not attained.

Existence

The Euler-Lagrange equation (1) has a solution in the following cases:

- $n = 2$, $R > 0$ (C.B., J. Fleckinger, F. de Thélin (2001))
- $n > 0$, R large (")
- $n = 1$, $N = 3$ ($p = 2, p^* = 5$) and any $R > 1$ (C.B. , R. Benguria (2002))

Theorem

Let $n = 1$. Then (1) has a solution for all $R > 1$. There is no solution if $R = 1$.

Sketch of the proof

Shooting method

$$(\rho^{N-p}(r)r^{N-1}|u'|^{p-2}u')' + \rho^N r^{N-1}u^{p^*-1} = 0 \quad r > 0,$$
$$u'(0) = 0 \text{ and } u(0) = \gamma.$$

For any $\gamma > 0$, $u(r)$ vanishes at a finite value $R(\gamma)$.
The function $\gamma \to R(\gamma)$ is continuous.
$R(\gamma) > 1$ and $R(\gamma) \to \infty$ as $\gamma \to 0$.

ODE techniques, Lyapunov estimates yield

$$R(\gamma) - 1 \to c_0\gamma^{-\left(\frac{N+1}{N-1}\right)^2} \text{ as } \gamma \to \infty.$$

Existence of minimizers, $n = 1$

Geodesic balls

Theorem

For every geodesic ball B_R containing the hemisphere ($R > 1$), $\tilde{S}_p(B_R)$ is attained. In addition $\tilde{S}_p(B_R) < S_p^*$.

Proof Minimizing sequence

$$u_\epsilon = \left(\epsilon + r^{p/(p-1)}\right)^{\frac{N-p}{p}} \underbrace{\psi}_{\text{cutoff function}}$$

More general domains

Corollary

$\tilde{S}_p(D)$ is attained for every domain $D \subset \mathbb{S}^N$ containing the hemisphere.

Consequently $\tilde{\Delta}_p u + u^{p^*-1} = 0, \; u > 0$ in $D, \quad u = 0$ on ∂D, has a solution.