Isoperimetric Bounds for Product Probability Measures

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joint work with

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The Isoperimetric Problem of Queen Dido
and its Mathematical Ramifications

28 May 2010

Isoperimetric Function

Let τ be a probability measure in $\mathbb{R}^N$.

- $\mathbf{I}_\tau(\cdot) : [0,1] \to \mathbb{R}^+$ is the **Isoperimetric Function** of τ:

$$\mathbf{I}_\tau(y) = \inf\{\tau^+(\partial A) \mid \tau(A) = y\}.$$

- For $A \subseteq \mathbb{R}^N$, with sufficiently smooth boundary, $\tau^+(\partial A)$ is the **boundary measure** of A:

$$\tau^+(\partial A) = \lim_{h \to 0^+} \frac{\tau(A_h \setminus A)}{h},$$

where $A_h = \{x \in \mathbb{R}^d \ : \ d(x, A) \leq h\}$.

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Product probability measures

Let τ be a probability measure on $\mathbb{R}$ with density:

$$d\tau(x) = f(x)dx = e^{\psi(x)}dx, \qquad x \in \mathbb{R},$$

We consider τ^N the **product probability measure** of τ:

$$d\tau^N(x) = \mathbf{f}(x)\,dx = \prod_{i=1}^{N} f(x_i)dx_i, \qquad x \in \mathbb{R}^N.$$

Isoperimetric Estimates

$\rightsquigarrow$ Can we estimate $\mathbf{I}_{\tau^N}(t)$ in terms of $\mathbf{I}_{\tau}(t)$?

- It always holds: $\mathbf{I}_{\tau^N}(t) \leq \mathbf{I}_{\tau}(t)$, $\forall t \in [0,1]$;

- Gaussian: $d\gamma(x) = e^{-x^2/2}/\sqrt{2\pi} \rightsquigarrow \mathbf{I}_{\gamma^N}(t) = \mathbf{I}_{\gamma}(t)$;
- Exponential: $d\nu(x) = \frac{1}{2}e^{-|x|} \rightsquigarrow \mathbf{I}_{\nu}(t)/2\sqrt{6} \leq \mathbf{I}_{\nu^N}(t) \leq \mathbf{I}_{\nu}(t)$.

[S.G. Bobkov, C. Houdré '97]

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- $\rightsquigarrow$ logistic measure

The logistic measure μ

$$d\mu(x) = \frac{e^x}{(1+e^x)^2}\, dx.$$

- μ is a C^2 log-concave measure in $\mathbb{R}$, with $\inf \psi'' = 0$;
- μ has Gaussian behaviour close to the origin and exponential tails;
- its distribution function $x(t)$ satisfies: $x' = x(1-x)$
- we look for C_μ s.t. $C_\mu I_\mu(t) \le I_{\mu^N}(t) \le I_\mu(t)$ $\forall t \in [0,1]$.

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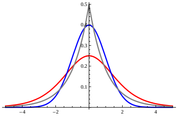

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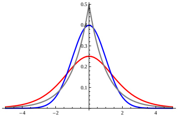

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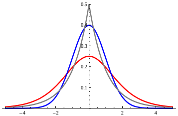

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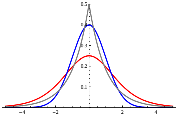

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Ingredients:

- the value of best costant in the Poincaré inequality: $\lambda_\mu = \frac{1}{4}$;
- an estimate by [F. Barthe, P. Cattiaux, C. Roberto, '07];

$\rightsquigarrow$ $\mathbf{I}_{\mu^N}(t) \ge \frac{C}{2} \, \mathbf{I}_\mu(t)$, with $C > 0.45$.

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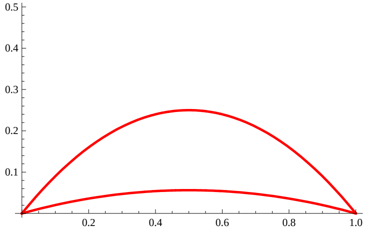

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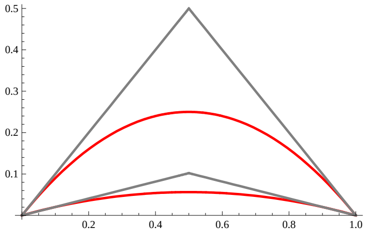

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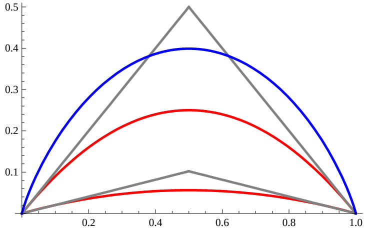

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Optimal sets

$A \subseteq \mathbb{R}^N$ is an **optimal set** for the measure τ^N if $\tau^N(A) = t$,

$$\mathbf{I}_{\tau^N}(t) = {\tau^N}^+(\partial A).$$

Consider μ^N, the N-product logistic measure:

- $N = 1 \rightsquigarrow$ half lines are optimal sets

  [S.G. Bobkov, '96]

- $N \geq 2 \rightsquigarrow$ can we guess that half spaces are optimal sets?

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Stationarity (I order condition)

$A \subset \mathbb{R}^d$ is a **stationary** set for τ: $d\tau = e^{\psi(x)}$ iff

$$\mathrm{H}_\psi(\partial A) = (N-1)\mathrm{H}(x) - \langle D\psi(x), \nu(x)\rangle\Big|_{\partial A} = \text{constant},$$

[C. Rosales, A. Cañete, V. Bayle, F. Morgan, '08]

Stationarity of half spaces

Let τ on $\mathbb{R}$: $d\tau(x) = e^{\psi(x)}dx$, with $\psi \in C^2(\mathbb{R})$ and $\tau \neq \gamma$. For $v \in \mathbb{S}^{N-1}$ let

$$H^N_{v,t} = \left\{ x \in \mathbb{R}^N \ : \ \langle x, v\rangle < t \right\}$$

The half space $H^N_{v,t}$ is stationary for τ^N if and only if:

- $H^N_{v,t}$ is a coordinate half space; **or**
- $v = \frac{1}{\sqrt{2}}(1, -1, 0, ..., 0)$ and ψ'' is $\sqrt{2}t$-periodic; **or**
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Stationarity of half spaces for the logistic measure

Half spaces which are stationary for the logistic measure are:

- the coordinate half spaces, and
- $H^N_{v,0}$ with $v = \frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, \ldots, 0)$.

[F. Barthe, CB, A. Colesanti]

Stability (II order condition)

$A \subset \mathbb{R}^d$ is a **stable** set for τ: $d\tau = e^{\psi(x)}$ iff A is stationary and for every function $u \in C_0^\infty(\partial A)$ such that $\int_{\partial A} u(x) f(x)\, dx = 0$

$$\int_{\partial A} f \left(|D_{\partial A} u|^2 - K^2 u^2 \right) d\mathscr{H}^{d-1} + \int_{\partial A} f\, u^2 \left\langle D^2 \psi \nu ; \nu \right\rangle d\mathscr{H}^{d-1} \geq 0,$$

where ν is the outer unit normal to ∂A.

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- If $H^N_{v,t}$ is a coordinate half space and $-\psi''(t) \leq \lambda_\tau$;
  Then the half space $H^N_{v,t}$ is stable for τ^N.
  Moreover:
- for $v = \frac{1}{\sqrt{2}} (\pm 1, \pm 1, 0, ..., 0)$,
  $H^N_{v,0}$ is stable if and only if so is $H^3_{v,0}$, for every $N \geq 3$.

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Stability of half spaces for the logistic measure

- $\forall N \geq 2$ coordinate half spaces with $|t| \geq 2\log(2+\sqrt{3})$;
- $N = 2$, $H^2_{v,0}$ with $v = \frac{1}{\sqrt{2}}(\pm 1, \pm 1)$,

  are stable for the logistic measure.
- $N \geq 3$ $v = \frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, ..., 0)$ half spaces

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$\rightsquigarrow$ $\{\mu^2\text{-stable half spaces}\} = \{x = \pm y\} \cup \{\text{some coordinate}\}$.

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[F. Barthe, CB, A. Colesanti]

Stability of half spaces for the logistic measure

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