Interpolating between torsional rigidity and principal frequency (joint work with Jesse Ratzkin)

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Carthage, 24th May 2010

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Torsional rigidity $P(D)$ is

$$\boxed{\frac{4}{P(D)} = \inf\left\{ \frac{\int_D |\nabla u(x)|^2 \, dx}{\left(\int_D u(x)\, dx\right)^2} : u > 0,\ u \in C_0^\infty(D) \right\}}$$

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We study a p-version of these minimisation problems for $1 \le p < \frac{2n}{n-2}$ ($p < \infty$ in dimension 2):

$$\boxed{\mathcal{C}_p(D) = \inf\left\{ \Phi_p(u) = \frac{\int_D |\nabla u(x)|^2 dx}{\left(\int_D u(x)^p dx\right)^{2/p}} : u > 0,\ u \in C_0^\infty(D) \right\}}$$

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- What can $\mathcal{C}_p(D)$ tell us about the common properties of $\lambda(D)$ and $P(D)$?
- cf. recent arXiv posting by Q. Dai, R. He and H. Hu
  Isoperimetric inequalities and sharp estimates for positive solutions of sublinear elliptic equations.

Euler-Lagrange equation

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This agrees with the pde for the torsion function ($p = 1$)

$$\Delta\phi + 2 = 0$$

and the pde for the first eigenfunction for the Laplacian ($p = 2$)

$$\Delta\phi + \lambda\phi = 0.$$

p-torsional rigidity

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$$\boxed{\mathcal{R}_p(D)\,\mathcal{C}_p(D) = 4}$$

Monotonicity

Theorem

If $1 \le p < q$ *then*

$$\mathrm{Vol}(D)^{2/p}\mathcal{C}_p(D) > \mathrm{Vol}(D)^{2/q}\mathcal{C}_q(D).$$

The inequality in this theorem is always strict.

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Theorem

$$\mathcal{C}_p(D) \geq \mathcal{C}_p(D^*)$$

for $p \geq 1$, with equality if and only if D is a ball to start with.

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$$\boxed{\phi_M^{2-p} \leq \frac{2\Lambda}{pA_p^2}\,R(D)^2} \quad \text{where } A_p = \int_0^1 \frac{dt}{\sqrt{1-t^p}}$$

Equality in the case of a strip / slab.

Proof that $\phi_M^{2-p} \leq \text{const} \times R(D)^2$

We follow Section 6.2.2 of Sperb's book. Payne's P-function

$$v(x) = |\nabla\phi(x)|^2 + \frac{2\Lambda}{p}\phi^p(x),$$

assumes its maximum at the point where ϕ assumes its maximum.

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$$R(D) \geq \delta_D(P) \geq \sqrt{\frac{p}{2\Lambda}}\phi_M^{(2-p)/2}A_p \quad \text{where } A_p = \int_0^1 \frac{dt}{\sqrt{1-t^p}}.$$

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- Thanks for listening!