Existence results in Prescribing Scalar Curvature on S^n

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Introduction

On $S^n = \{x \in \mathbb{R}^{n+1}, \ \|x\| = 1\}$, the standard metric $g_0 = \sum_{i=1}^{n+1} dx_i^2$ has constant scalar curvature $R(g_0) = n(n-1)$. A conformally metric g to g_0 is usually written as $g = u^{\frac{4}{n-2}} g_0$, where $0 < u \in C^2(S^n), \ n \geq 3$.
The Scalar curvature function $R(g)$ of the metric g is given by the differential equation

$$-L_{g_0} u := -\frac{4(n-1)}{n-2} \Delta_{g_0} u + n(n-1)u = R(g) u^{\frac{n+2}{n-2}} \quad on S^n. \quad (F1)$$

Here, $-l_{g_0}$ is the conformal Laplacien operator of (S^n, g_0).

Introduction

A well known problem in the conformal geometry is the Yamabe problem or more generally, the prescribed scalar curvature problem. The problem is to decide which function on S^n can be the scalar curvature of a conformal metric. More precisely, given a smooth function

$$K : S^n \longrightarrow \mathbb{R},$$

can one change the original metric g_0 conformally into a new metric g such that $K(x)$ is the scalar curvature of g.

Introduction

According to the formula $(F1)$, the problem is equivalent to solve the following nonlinear P.D.E.

$$(1)\begin{cases} -L_{g_0}u = Ku^{\frac{n+2}{n-2}}, & \\ u > 0, & \text{on } S^n. \end{cases}$$

Introduction

The scalar curvature problem has been continuing to be one of major subjects in nonlinear P. D. E's. The literature for the existence of solutions of the equation (1) is considerably bigger and many authors "was studied this problem, we can cite Aubin, Bahri, Bresis, Chang, Coron, Schoen, Y.Li, Zhang and others.

Introduction

Equation (1) has a variational structure. A natural space to look in for solutions is $H^1(S^n)$. We recall that by regularity theorem, a week solution of (1) is indeed a smooth solution. Due to the noncompactness of the injection of $H^1(S^n)$ into $L^{\frac{2n}{n-2}}(S^n)$, the Euler functional associated to (1) does not satisfy the Palais-Smale condition, which leads to the failure of the standard critical point theory.

Introduction

There are many methods to study this problem we can cite :

1. The minimization methods

   T. Aubin, Brezis, Chang, SChoen.

2. Subcritical approximations and blow-up analysis

   Schoen, Y. Li.

3. Critical points at infinity method

   Aubin, Bahri, Brezis, Coron.

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Known results

One group of existence results have been obtained under the following non-degeneracy condition

(nd) $\Delta K(y) \neq 0$ whenever $\nabla K(y) = 0$

A typical result says that a solution of (1) exists provided K is a Morse positive function satisfying (nd) condition and

$$(F1) \qquad \sum_{y \in \mathcal{K}^+} (-1)^{n - ind(K,y)} \neq 1,$$

Where $ind(K, y)$ is the Morse index of K at y and

$$\mathcal{K}^+ = \{y \in S^n, \nabla K(y) = 0 \; and \; -\Delta K(y) > 0\}$$

Known results

- For n=3, this result have been given first by Bahri-Coron 91' and later by Chang-Gursky-Yang 93' and Schoen-Zhang 96'.
- For $n \geq 4$ the result have been given by Chang-Yang 91'. Under a perturbative condition, that is for K close to a constant function.

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New results

A natural question which arises when looking to the above results, is what happens if the total sum in $(F1)$ is equal to 1, but a partial one is not ? Under which condition can one use this partial sum to derive an existence result ?
Our aim in this talk is to give a partial answer to this question and to prove new existence criterium which provides existence results for a dense subset of Morse positive functions satisfying (nd) and generalizes the index-count formula of type $(F1)$.
To state our results, we set the following notations, let

New results

$$\mathcal{K} = \{y \in S^n,\ \nabla_g K(y) = 0\},$$

$$\mathcal{K}^+ = \{y \in \mathcal{K};\ -\Delta_g K(y) > 0\}. \quad (1)$$

We say that an integer $k \in (A_1)$ if it satisfies the following :

(A1) For each $z \in \mathcal{K}^+$, such that $3 - \text{ind}(K, z) = k + 1$ and for each $y \in \mathcal{K}^+$ such that $3 - \text{ind}(K, y) \leq k$, we have :

$$\frac{1}{K(z)^{\frac{1}{2}}} > \frac{1}{K(y_0)^{\frac{1}{2}}} + \frac{1}{K(y)^{\frac{1}{2}}},$$

Here y_0 is an absolute maximum of K on S^3.

where $ind(K, y)$ denotes the Morse index of K at y.

Our first main result is the following

New results

Theorem 1. (R. Ben mahmoud - C, Annales de l'Institut Fourier 2010)

Let K be a Morse positive function on S^3 satisfying (nd) condition, if

$$\underset{k \in (\mathbf{A_1})}{\text{Max}} \left| 1 - \sum_{\substack{y \in \mathcal{K}^+ \\ 3 - \text{ ind } (K, y) \leq k}} (-1)^{3 - \text{ind}(K,y)} \right| \neq 0$$

then there exist a solution to problem (1).

New results

Please observe that, any integer $k \geq 2$ satisfies condition (A_1). It follows that the celebrated result of Bahri-Coron is a corollary of our theorem. Namely we have

Corollaire 1. Bahri-Coron 1991

Let K be a Morse positive function on S^3 satisfying (nd) condition if,

$$\sum_{y \in \mathcal{K}^+} (-1)^{3-\text{ind}(K,y)} \neq 1,$$

then the problem (1) has at least one solution.

New results

We point out that the main new contribution of theorem 1.1 is that we address here the case where the total sum in the above corollary equals to 1, but a partial one is not equal to 1. The main issue being the possibility to use such information to prove existence results.

Remarque

The used method in the proof of theorem 1.1 enables us to provide an upper-bound of the Morse indices of the obtained solutions.

New results

Now, we give another kind of existence result, which is not based on the index-count formula.

Theorem 2. (R. Ben mahmoud - C, Annales de l'Institut Fourier 2010)

Assume that the function K is a Morse function satisfies (nd) and satisfies the following :
There exists $\widetilde{y} \in \mathcal{K}^+$, $ind(K, \widetilde{y}) = 1$ such that :

$$K(\widetilde{y}) \geq K(y), \quad \forall y \in \mathcal{K}^+ \text{ such that } ind(K, y) = 2,$$

then(1) has at least one solution.

New results

Question :

One question consists in removing the condition $(A1)$ from theorem 1. This enables us to prove the following result :

$$\text{if } \mathcal{K}^+ \backslash \{y_0\} \neq \varnothing, \text{ then } (1) \text{ has a solution.}$$

Here y_0 is an absolute maximum of K on S^3.

New results

In the second part of this work, we consider the case $n \geq 4$ and K close to a constant, i.e. K of the form $K = 1 + \varepsilon K_0$, for $K_0 \in C^2(S^n)$ and $\mid \varepsilon \mid$ small. So we are reduced to study the problem on S^n,

$$L^n_{g_0} u = (1 + \varepsilon K_0) u^{\frac{n+2}{n-2}}, \quad u > 0 \qquad (P_\varepsilon)$$

Our aim is to tackle the problem (P_ε) using an another approach completely different from the ones used by A. Cang and P. Yang and to extend the existence result subject of theorem 1.1 for any dimension $n \geq 4$. We introduce the following notation.

We say that an integer $k \in (A2)$ if it satisfies the following

For any $y \in \mathcal{K}^+$, we have $n - ind(K, y) \neq k + 1$, (A2)

New results

We then have the following result

Theorem 3. (C, 2010)

Let $n \geq 4$. Assume that the function K is a Morse function satisfies (nd). If,

$$\max_{k \in (A2)} \left| 1 - \sum_{y \in \mathcal{K}^+,\ n - ind(K,y) \leq k.\ .} (-1)^{n - ind(K,y)} \right| \neq 0,$$

then , for $\mid \varepsilon \mid$ sufficiently small there exists a solution to the problem (P_ε).

New results

Please observe that, any integer $k \geq n$ satisfies condition $(A2)$. Thus, as consequence of the above theorem we have the following corollary

Corollaire 2. (Chang-Yang, 1991)

Let $n \geq 4$. Under the assumption (nd), if,

$$\sum_{y \in \mathcal{K}^+} (-1)^{n-ind(K,y)} \neq 1,$$

Then for $\mid \varepsilon \mid$ sufficiently small, (P_ε) has at least one solution.