Isocapacitary vs. isoperimetric methods in eigenvalue problems for the Laplacian on manifolds

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Carthage, May 2010

1. A.C. & V.Maz'ya Bounds for eigenfunctions of the Laplacian on noncompact Riemannian manifolds, preprint.
2. A.C. & V.Maz'ya On the discreteness of the spectrum of noncompact Riemannian manifolds, preprint

Let M be an n-dimensional Riemannian manifold (of class C^1) such that

$$\mathcal{H}^n(M) < \infty.$$

Here, $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on M, namely, the volume measure on M induced by its Riemannian metric.

Problem: estimates for eigenfunctions of the Laplacian on M.
Weak formulation: a function $u \in W^{1,2}(M)$ is an eigenfunction of the Laplacian associated with the eigenvalue γ if

$$\int_M \nabla u \cdot \nabla \Phi \, d\mathcal{H}^n(x) = \gamma \int_M u\Phi \, d\mathcal{H}^n(x) \tag{1}$$

for every $\Phi \in W^{1,2}(M)$.

If M is complete, then (1) is equivalent to

$$-\Delta u = \gamma u \qquad \text{on } M. \tag{2}$$

If M is an open subset of a Riemannian manifold, in particular of $\mathbb{R}^n$, then (1) is the weak formulation of the Neumann problem

$$\begin{cases} -\Delta u = \gamma u & \text{on } M \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial M \end{cases} \tag{3}$$

Case M compact.
The eigenvalue problem for the Laplacian has been extensively studied.

By the classical Rellich's Lemma , the compactness of the embedding

$$W^{1,2}(M) \to L^2(M)$$

is equivalent to the discreteness of the spectrum of the Laplacian on M.

Bounds for eigenfunctions in $L^q(M)$, $q > 2$, and $L^\infty(M)$ follow via local bounds, owing to the compactness of M.

Pb.: noncompact M.

Much less seems to be known.

Not even the existence of eigenfunctions is guaranteed.

Major problem: the embedding $W^{1,2}(M) \to L^2(M)$ need not be compact.

Example 1.

$$M = \Omega$$

an open subset of $\mathbb{R}^n$ endowed with the Eulcidean metric.
The eigenvalue problem (2) turns into the Neumann problem

$$\begin{cases} -\Delta u = \gamma u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \,. \end{cases}$$

The point here is that no regularity on $\partial\Omega$ is (a priori) assumed.
Contributions in [B.Simon], [Burenkov-Davies].

Example 2.
A noncompact manifold of revolution in $\mathbb{R}^n$,

$$M = \{(r, \omega) : r \in [0, \infty), \omega \in \mathbb{S}^{n-1}\},$$

with metric (in polar coordinates) given by

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2 \,. \tag{4}$$

Here, $d\omega^2$ stands for the standard metric on $\mathbb{S}^{n-1}$, and $\varphi : [0, L) \to [0, \infty)$ is a smooth function such that $\varphi(r) > 0$ for $r \in (0, L)$, and

$$\varphi(0) = 0 \,, \qquad \text{and} \qquad \varphi'(0) = 1 \,.$$

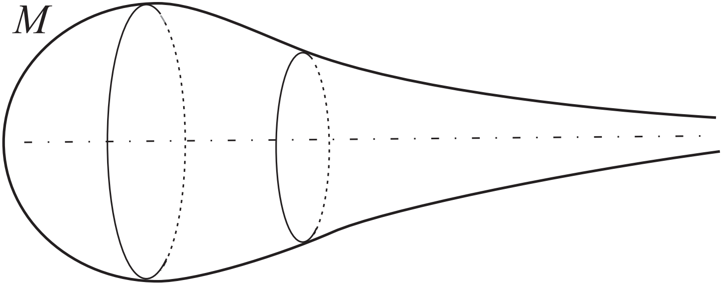

FIGURE: A manifold of revolution

Example 3.
Manifolds of Courant-Hilbert type.

M contains a sequence of mushroom-shaped submanifolds .

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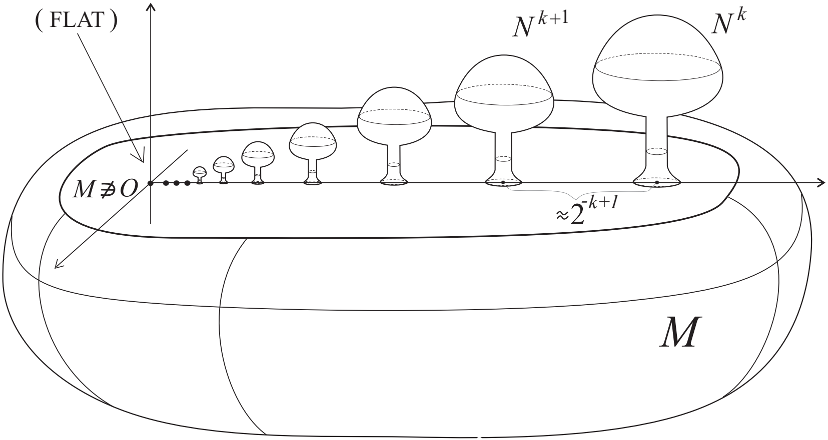
(FLAT)
N^{k+1}
N^k
$M \not\ni O$
$\approx 2^{-k+1}$
M

Qualitative and quantitative properties of eigenvalues and eigenfunctions depend on the geometry of M.

A possible description of the geometry of the manifold M is via the

isoperimetric function λ_M of M.

The use of isoperimetric inequalities in the study of Dirichlet eigenvalue problems on domains of $\mathbb{R}^n$ is classical: [Faber, 1923], [Krahn, 1925], [Payne-Pólya-Weiberger, 1956], [Chiti, 1983], [Ashbaugh-Benguria, 1992], [Nadirashvili, 1995] ...

An alternate approach, exploiting the

isocapacitary function ν_M of M,

is more effective in dealing with manifolds having an irregular geometry (in particular, Neumann eigenvalue problems on irregular domains in $\mathbb{R}^n$).

Classical isoperimetric inequality [De Giorgi]

$$\mathcal{H}^{n-1}(\partial^* E) \geq n\omega_n^{1/n}|E|^{1/n'} \qquad \forall E \subset \mathbb{R}^n .$$

Here:

- $\partial^* E$ stands for the essential boundary of E,
- $|E| = \mathcal{H}^n(E)$, the Lebesgue measure of E,
- $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (the surface area).

In other words,

the ball has the least surface area among sets of fixed volume.

In general the isoperimetric function $\lambda_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty)$ of M (introduced by V.G.Maz'ya) is defined as

$$\lambda_M(s) = \inf\{\mathcal{H}^{n-1}(\partial^* E) : s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2\}\,, \tag{5}$$

for $s \in [0, \mathcal{H}^n(M)/2]$.

Isoperimetric inequality on M:

$$\mathcal{H}^{n-1}(\partial^* E) \geq \lambda_M(\mathcal{H}^n(E)) \quad \forall E \subset M, \mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2.$$

The geometry of M is related to λ_M, and, in particular, to its asymptotic behavior at 0. For instance, if M is compact, then

$$\lambda_M(s) \approx s^{1/n'} \qquad \text{as } s \to 0.$$

Here, $f \approx g$ means that $\exists\ c, k > 0$ such that

$$cg(cs) \leq f(s) \leq kg(ks).$$

Moreover, $n' = \dfrac{n}{n-1}$.

Approach by isocapacitary inequalities.
Standard capacity of $E \subset M$:

$$C(E) = \inf \Big\{ \int_M |\nabla u|^2 \, dx : u \in W^{1,2}(M), \\ "u \geq 1" \text{ in } E, \text{and } u \text{ has compact support} \Big\}.$$

Capacity of a condenser $(E; G)$, $E \subset G \subset M$:

$$C(E;G) = \inf \Big\{ \int_M |\nabla u|^2 \, dx : u \in W^{1,2}(M), \\ "u \geq 1" \text{ in } E \ "u \leq 0" \text{ in } M \setminus G \Big\}.$$

Isocapacitary function (introduced by V.G.Maz'ya)

$$\nu_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty)$$

$$\nu_M(s) = \inf\{C(E,G) : E \subset G \subset M,\ s \leq \mathcal{H}^n(E) \text{ and } \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2\}$$

for $s \in [0, \mathcal{H}^n(M)/2]$.

Isocapacitary inequality:

$$C(E,G) \geq \nu_M(\mathcal{H}^n(E)) \quad \forall\ E \subset G \subset M,\ \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2.$$

If M is compact and $n \geq 3$, then

$$\nu_M(s) \approx s^{\frac{n-2}{n}} \qquad \text{as } s \to 0.$$

The isoperimetric function and the isocapacitary function of a manifold M are related by

$$\frac{1}{\nu_M(s)} \leq \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2). \qquad (6)$$

A reverse estimate does not hold in general.
Roughly speaking, a reverse estimate only holds when the geometry of M is sufficiently regular.

Both the conditions in terms of ν_M, and those in terms of λ_M, for eigenfunction estimates in $L^q(M)$ or $L^\infty(M)$ to be presented are sharp in the class of manifolds M with prescribed asymptotic behavior of ν_M and λ_M at 0.

Each one of these approaches has its own advantages.

The isoperimetric function λ_M has a transparent geometric character, and it is usually easier to investigate.

The isocapacitary function ν_M is in a sense more appropriate: it not only implies the results involving λ_M, but leads to finer conclusions in general. Typically, this is the case when manifolds with complicated geometric configurations are taken into account.

Estimates for eigenfunctions.
If u is an eigenfunction of the Laplacian, then, by definition, $u \in W^{1,2}(M)$. Hence, trivially, $u \in L^2(M)$.

Problem: given $q \in (2, \infty]$, find conditions on M ensuring that any eigenfunction u of the Laplacian on M belongs to $L^q(M)$.

Theorem 1: L^q bounds for eigenfunctions

Assume that

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0\,. \qquad (7)$$

Then for any $q \in (2, \infty)$ there exists a constant C such that

$$\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)} \qquad (8)$$

for every eigenfunction u of the Laplacian on M.

The assumption

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0 \tag{9}$$

is essentially minimal in Theorem 1.

Theorem 2: Sharpness of condition (9)

For any $n \geq 2$ and $q \in (2, \infty]$, there exists an n-dimensional Riemannian manifold M such that

$$\nu_M(s) \approx s \qquad \text{near } 0, \tag{10}$$

and the Laplacian on M has an eigenfunction $u \notin L^q(M)$.

Conditions in terms of the isoperimetric function for L^q bounds for eigenfunctions can be derived via Theorem 2.

Corollary 2

Assume that

$$\lim_{s \to 0} \frac{s}{\lambda_M(s)} = 0 \,. \tag{11}$$

Then for any $q \in (2, \infty)$ there exists a constant C such that

$$\|u\|_{L^q(M)} \leq C \|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M.

Assumption (12) is minimal in the same sense as the analogous assumption in terms of ν_M.

Estimate for the growth of constant in the $L^q(M)$ bound for eigenfunctions in terms of the eigenvalue.

Proposition

Assume that

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0\,. \tag{12}$$

Define

$$\Theta(s) = \sup_{r\in(0,s)} \frac{r}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then $\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)}$ for any $q \in (2,\infty)$ and for every eigenfunction u of the Laplacian on M associated with the eigenvalue γ, where

$$C(\nu_M, q, \gamma) = \frac{C_1}{\left(\Theta^{-1}(C_2/\gamma)\right)^{\frac{1}{2}-\frac{1}{q}}}\,,$$

and $C_1 = C_1(q, \mathcal{H}^n(M))$ and $C_2 = C_2(q, \mathcal{H}^n(M))$.

Example.

Assume that there exists $\beta \in [(n-2)/n, 1)$ such that

$$\nu_M(s) \geq C s^{\beta}.$$

Then there exists a constant $C = C(q, \mathcal{H}^n(M))$ such that

$$\|u\|_{L^q(M)} \leq C \gamma^{\frac{q-2}{2q(1-\beta)}} \|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M associated with the eigenvalue γ.

Consider now the case when $q = \infty$, namely the problem of the boundedness of the eigenfunctions.

Theorem 3: boundedness of eigenfunctions

Assume that

$$\int_0 \frac{ds}{\nu_M(s)} < \infty \,. \tag{13}$$

Then there exists a constant C such that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{L^2(M)} \tag{14}$$

for every eigenfunction u of the Laplacian on M.

The condition

$$\int_0 \frac{ds}{\nu_M(s)} < \infty \tag{15}$$

is essentially sharp in Theorem 4.
This is the content of the next result.
Recall that $f \in \Delta_2$ near 0 if there exist constants c and s_0 such that

$$f(2s) \leq cf(s) \qquad \text{if } 0 < s \leq s_0. \tag{16}$$

Theorem 4: sharpness of condition (15)

Let ν be a non-decreasing function, vanishing only at 0, such that

$$\lim_{s\to 0} \frac{s}{\nu(s)} = 0\,, \tag{17}$$

but

$$\int_0 \frac{ds}{\nu(s)} = \infty\,. \tag{18}$$

Assume in addition that $\nu \in \Delta_2$ near 0 and

$$\frac{\nu(s)}{s^{\frac{n-2}{n}}} \quad \text{is equivalent to a non-decreasing function near } 0, \tag{19}$$

for some $n \geq 3$. Then, there exists an n-dimensional Riemannian manifold M fulfilling

$$\nu_M(s) \approx \nu(s) \qquad \text{as } s \to 0, \tag{20}$$

and such that the Laplacian on M has an unbounded eigenfunction.

Assumption (19) is consistent with the fact that $\nu_M(s) \approx s^{\frac{n-2}{n}}$ near 0 if the geometry of M is nice (e.g. M compact), and that $\nu_M(s) \to 0$ faster than $s^{\frac{n-2}{n}}$ otherwise.

Owing to the inequality

$$\frac{1}{\nu_M(s)} \leq \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2),$$

Theorem 4 has the following corollary in terms of isoperimetric inequalities.

Corollary 3

Assume that

$$\int_0 \frac{s}{\lambda_M(s)^2}\, ds < \infty\,. \tag{21}$$

Then there exists a constant C such that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{L^2(M)} \tag{22}$$

for every eigenfunction u of the Laplacian on M.

Assumption (21) is sharp in the same sense as the analogous assumption in terms of ν_M.

Estimate for the growth of constant in the $L^\infty(M)$ bound for eigenfunctions in terms of the eigenvalue.

Proposition

Assume that

$$\int_0 \frac{ds}{\nu_M(s)} < \infty.$$

Define

$$\Xi(s) = \int_0^s \frac{dr}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then $\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)}$ for every eigenfunction u of the Laplacian on M associated with the eigenvalue γ, where

$$C(\nu_M, \gamma) = \frac{C_1}{\left(\Xi^{-1}(C_2/\gamma)\right)^{\frac{1}{2}}},$$

and C_1 and C_2 are absolute constants.

Example.

Assume that there exists $\beta \in [(n-2)/n, 1)$ such that

$$\nu_M(s) \geq C s^\beta.$$

Then there exists an absolute constant C such that

$$\|u\|_{L^\infty(M)} \leq C \gamma^{\frac{1}{2(1-\beta)}} \|u\|_{L^2(M)}$$

for every eigenfunction u of the Laplacian on M associated with the eigenvalue γ.

Pb.: Discreteness of the spectrum of the Laplacian on M.
Consider the semi-definite self-adjoint Laplace operator Δ on the Hilbert space $L^2(M)$ associated with the bilinear form
$a : W^{1,2}(M) \times W^{1,2}(M) \to \mathbb{R}$ defined as

$$a(u, v) = \int_M \nabla u \cdot \nabla v \, d\mathcal{H}^n(x) \tag{23}$$

for $u, v \in W^{1,2}(M)$.
• When $\overline{C_0^\infty(M)} = W^{1,2}(M)$, the operator Δ agrees with the Friedrichs extension of the classical Laplace operator. This is the case, for instance, if M is complete, and, in particular, if M is compact.
• When M is an open subset of $\mathbb{R}^n$, or, more generally, of a Riemannian manifold, then Δ corresponds to the Neumann Laplacian on M.

A necessary and sufficient condition for the discreteness of the spectrum of Δ can be given in terms of the isocapacitary function of M.

Theorem 5: Discreteness of the spectrum of Δ

The spectrum of the Laplacian on M is discrete if and only if

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0 \,. \qquad (24)$$

Condition (24) agrees with that ensuring $L^q(M)$ bounds for eigenfunctions.

The proof of Theorem 5 relies upon the following characterization of the compactness of the embedding

$$W^{1,2}(M) \to L^2(M). \tag{25}$$

Theorem 6: Compactness of the embedding (25)

The embedding

$$W^{1,2}(M) \to L^2(M)$$

is compact if and only if

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0\,.$$

As a consequence of Theorem 5, the following sufficient condition in terms of the isoperimetric function of M holds.

Corollary 4

Assume that

$$\lim_{s \to 0} \frac{s}{\lambda_M(s)} = 0\,.$$

Then the spectrum of the Laplacian on M is discrete.

Example 4

Manifold of revolution, with metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2 \tag{26}$$

and $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\varphi(r) = e^{-r^\alpha} \qquad \text{for large } r. \tag{27}$$

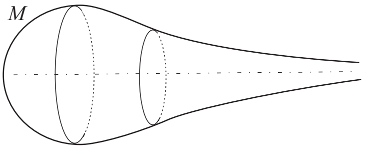

FIGURE: A manifold of revolution

The larger is α, the better is M.
One can show that

$$\lambda_M(s) \approx s\big(\log(1/s)\big)^{1-1/\alpha} \qquad \text{near } 0,$$

and

$$\nu_M(s) \approx \left(\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}\right)^{-1} \approx s\big(\log(1/s)\big)^{2-2/\alpha} \qquad \text{near } 0.$$

The criteria involving λ_M tell us that all eigenfunctions of the Laplacian on M belong to $L^q(M)$ for $q < \infty$ if

$$\alpha > 1, \tag{28}$$

and to $L^\infty(M)$ if

$$\alpha > 2. \tag{29}$$

The same conclusions follow via the criteria involving ν_M.

Moreover, if $\alpha > 1$, then there exist constants $C_1 = C_1(q)$ and $C_2 = C_2(q)$ such that

$$\|u\|_{L^q(M)} \le C_1 e^{C_2 \gamma^{\frac{\alpha}{2\alpha-2}}} \|u\|_{L^2(M)}$$

for any eigenfunction u of the Laplacian associated with the eigenvalue γ.

If $\alpha > 2$, then there exist absolute constants C_1 and C_2 such that

$$\|u\|_{L^\infty(M)} \le C_1 e^{C_2 \gamma^{\frac{\alpha}{\alpha-2}}} \|u\|_{L^2(M)}$$

for any eigenfunction u associated with γ.
The spectrum of the Laplacian on M is discrete if and only if

$$\alpha > 1$$

.

Example 5

Manifolds with clustering submanifolds.

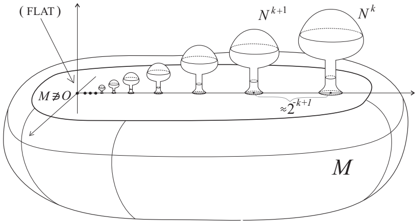

FIGURE: A manifold with a family of clustering submanifolds

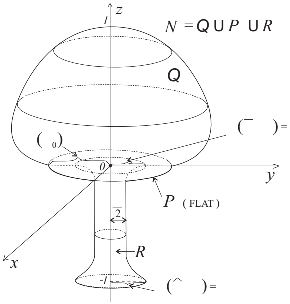

FIGURE: An auxiliary submanifold

In the sequence of mushrooms, the width of the heads and the length of the necks decay like 2^{-k}, the width of the neck decays like $\sigma(2^{-k})$ as $k \to \infty$, where

$$\lim_{s \to 0} \frac{\sigma(s)}{s} = 0.$$

Assume, for instance, that $b > 1$ and

$$\sigma(s) = s^b \qquad \text{for } s > 0.$$

Then the criterion involving λ_M ensures that all eigenfunctions of the Laplacian on M are bounded provided that

$$b < 2.$$

The criterion involving ν_M yields the boundedness of eigenfunctions under the weaker assumption that

$$b < 3$$

.

By the use of ν_M we also get that if $b < 3$, then there exists a constant $C = C(q)$ such that

$$\|u\|_{L^q(M)} \leq C\gamma^{\frac{q-2}{q(3-b)}}\|u\|_{L^2(M)}$$

for every $q \in (2, \infty]$ and for any eigenfunction u of the Laplacian associated with the eigenvalue γ.

Moreover, the characterization via ν_M implies that the spectrum of the Laplacian on M is discrete if and only if

$$b < 3$$

.

The use of λ_M tells us that spectrum of the Laplacian is discrete for

$$b < 2$$

only.

This example shows that the use of the isocapacitary function can actually lead to sharper conclusions than those obtained via the isoperimetric function.