Laplacian on Riemannian manifolds

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Lecture 1: Introduction, basic results and examples

Lecture 2: The case of the negatively curved compact manifolds

Lecture 3: Estimates on the conformal class

Lecture 4: The spectrum of submanifolds of the euclidean space

The 3 last lectures are independant.

Lecture 1: Introduction, basic results and examples

- Let (M, g) be a smooth, connected and C^∞ Riemannian manifold with boundary ∂M.
- The boundary is a Riemannian manifold with induced metric $g_{|\partial M}$.
- We suppose ∂M to be smooth.

- For a function $f \in C^2(M)$, we define the Laplace operator or Laplacian by

$$\Delta f = \delta df = -div\ gradf.$$

- d is the exterior derivative and δ the adjoint of d with respect to the usual L^2-inner product

$$(f, h) = \int_M f\ h\ dV$$

- dV denotes the volume form on (M, g).

- In local coordinates $\{x_i\}$, the Laplacian reads

$$\Delta f = -\frac{1}{\sqrt{det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j}(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_i} f).$$

- In particular, in the Euclidean case, we recover the usual expression

$$\Delta f = -\sum_{j} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f.$$

- Let $f \in C^2(M)$ and $h \in C^1(M)$ such that $h\ df$ has compact support in M. Then we have Green's Formula

$$(\Delta f, h) = \int_M \langle df, dh \rangle dV - \int_{\partial M} h \frac{df}{dn} dA$$

- $\frac{df}{dn}$ denotes the derivative of f in the direction of the outward unit normal vector field n on ∂M
- dA is the volume form on ∂M.

In particular, if one of the following conditions $\partial M = \emptyset$, $h_{|\partial M} = 0$ or $(\frac{df}{dn})_{|\partial M} = 0$ is satisfied, then we have the relation

$$(\Delta f, h) = (df, dh),$$

that is

$$\int_M \Delta f \ h \ dV = \int_M \langle df, dh \rangle dV$$

In the sequel, we will study the following eigenvalue problems when M is compact:

- Closed Problem:

$$\Delta f = \lambda f \;\; in \;\; M; \;\; \partial M = \emptyset;$$

- Dirichlet Problem

$$\Delta f = \lambda f \;\; in \;\; M; \;\; f_{|\partial M=0};$$

- Neumann Problem:

$$\Delta f = \lambda f \;\; in \;\; M; \;\; (\frac{df}{dn})_{|\partial M} = 0.$$

Theorem

Let M be a compact manifold with boundary ∂M (eventually empty), and consider one of the above mentioned eigenvalue problems. Then:

- *The set of eigenvalue consists of an infinite sequence* $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$, *where* 0 *is not an eigenvalue in the Dirichlet problem;*
- *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are* $L^2(M)$*-orthogonal;*
- *The direct sum of the eigenspaces* $E(\lambda_i)$ *is dense in* $L^2(M)$ *for the* L^2*-norm. Futhermore, each eigenfunction is* C^∞*-smooth and analytic.*

Remark

The Laplace operator depends only on the given Riemannian metric. If

$$F : (M, g) \longrightarrow (N, h)$$

is an isometry, then (M, g) *and* (N, h) *have the same spectrum, and if* f *is an eigenfunction on* (N, h)*, then* $f \circ F$ *is an eigenfunction on* (M, g) *for the same eigenvalue.*

Weyl law: If (M, g) is a compact Riemannian manifold of dimension n, then

$$\lambda_k(M, g) \sim \frac{(2\pi)^2}{\omega_n^{2/n}} \left(\frac{k}{Vol(M, g)}\right)^{2/n} \tag{1}$$

as $k \to \infty$, where ω_n denotes the volume of the unit ball of $\mathbb{R}^n$.

In these lectures, I will investigate the question "can λ_k (and in particular λ_1) be very large or very small?". The question seems trivial or naive at the first view, but it is not, and I will try to explain that partial answers to it are closely related to geometric properties of the considered Riemannian manifold.

- Of course, there is a trivial way to produce arbitrarily small or large eigenvalues:
- Take any Riemannian manifold (M, g). For any constant $c > 0$, $\lambda_k(c^2 g) = \frac{1}{c^2}\lambda_k(g)$ and an homothety produce small or large eigenvalues.
- So, we have to introduce some normalizations, in order to avoid the trivial deformation of the metric given by an homothety.

Most of the time, these normalizations are of the type "volume is constant" or "curvature and diameter are bounded".

Main goals:

Question 1:

- Try to find constants a_k and b_k depending on geometrical invariants such that, given a compact Riemannian manifold (M, g), we have

$$a_k(g) \leq \lambda_k(M, g) \leq b_k(g).$$

- There are a lot of possible geometric invariants:
- invariants depending on upper or lower bounds of the curvature of (M, g);

- on upper or lower bounds of the volume or of the diameter;
- on a lower bound of the injectivity radius of (M, g).
- This will appear concretely during the lecture.

Question 2: Are the bounds a_k and b_k in some sense optimal?

We can give different meaning to the word "optimal":

- For example, to see that a_k (or b_k) is optimal, we can try to construct a manifold (M, g) for which $\lambda_k(M, g) = a_k$ (or $\lambda_k(M, g) = b_k$).
- If this is not possible we can do a little less:
- to construct a *family* (M_n, g_n) of manifold with $\lambda_k(M_n, g_n)$ arbitrarily close to $a_k(g_n)$ (or $b_k(g_n)$) as $n \longrightarrow \infty$, or such that the ratio $\frac{\lambda_k(M_n, g_n)}{a_k(g_n)} \longrightarrow 1$ as $n \longrightarrow \infty$.

Note that, concretely, this is difficult, and we can hope to realize such a construction only for small k, in particular $k = 1$.

Question 3: Describe all manifolds (M, g) such that $\lambda_k(M, g) = a_k$. Again, this is difficult and you may hope to do this only for small k.

To investigate the Laplace equation $\Delta f = \lambda f$ is a priori a problem of analysis. To introduce some geometry on it, it is very relevant to look at the variational characterization of the spectrum.

- Rayleigh quotient

- If a function f lies in $H^1(M)$ in the closed and Neumann problems, and on $H_0^1(M)$ in the Dirichlet problem, the Rayleigh quotient of f is

$$R(f) = \frac{\int_M |df|^2 dV}{\int_M f^2 dV} = \frac{(df, df)}{(f, f)}.$$

- If f is an eigenfunction for λ_k, we have

$$R(f) = \frac{\int_M |df|^2 dV}{\int_M f^2 dV} = \frac{\int_M \Delta f \;\, f dV}{\int_M f^2 dV} = \lambda_k.$$

- Variational characterization of the spectrum:
- Let us consider one of the 3 eigenvalues problems. We denote by $\{f_i\}$ an orthonormal system of eigenfunctions associated to the eigenvalues $\{\lambda_i\}$.
- We have

$$\lambda_k = inf\{R(u) : u \neq 0; u \perp f_0, .., f_{k-1}\}$$

where $u \in H^1(M)$ ($u \in H_0^1(M)$ for the Dirichlet eigenvalue problem) and $R(u) = \lambda_k$ if and only if u is an eigenfunction for λ_k.

In particular, for a compact Riemannian manifold without boundary, we have the classical fact

$$\lambda_1(M,g) = inf\{R(u) : u \neq 0; \int_M udV = 0\}.$$

- Min-Max: we have

$$\lambda_k = \inf_{V_k} sup\{R(u) : u \neq 0,\ u \in V_k\}$$

where V_k runs through $k+1$-dimensional subspaces of $H^1(M)$ (k-dimensional subspaces of $H^1_0(M)$ for the Dirichlet eigenvalue problem).

- In particular, we have the very useful fact: for any given $(k+1)$ dimensional vector subspace V of $H^1(M)$,

$$\lambda_k(M,g) \leq sup\{R(u) : u \neq 0,\ u \in V\}.$$

- A special situation is if V_k is generated by $k+1$ disjointly supported functions $f_1, ..., f_{k+1}$:
- because

$$sup\{R(u) : u \neq 0,\ \ u \in V_k\} =$$

$$sup\{R(f_i) : i = 1, ..., k+1\},$$

the estimation becomes particularly easy to do.

- We can see already two advantages to this variational characterisation of the spectrum.
- We don't need to work with solutions of the Laplace equation, but only with "test functions", which is easier.
- We have only to control one derivative of the test function, and not two, as in the case of the Laplace equation.

Example: Monotonicity in the Dirichlet problem.

- Let $\Omega_1 \subset \Omega_2 \subset (M, g)$, two domains of the same dimension n of a Riemannian manifold (M, g). Let us suppose that Ω_1 and Ω_2 are both compact connected manifolds with boundary.
- If we consider the Dirichlet eigenvalue problem for Ω_1 and Ω_2 with the induced metric, then for each k
$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$$
with equality if and only if $\Omega_1 = \Omega_2$.

As a consequence, we have the following: if M is a compact manifold without boundary, and if Ω_1,...,Ω_{k+1} are domains in M with disjoint interiors, then

$$\lambda_k(M, g) \leq max(\mu_1(\Omega_1), ..., \mu_1(\Omega_{k+1})),$$

where $\mu_1(\Omega)$ denotes the first eigenvalue of Ω for the Dirichlet problem.

The Cheeger's dumbbell. The idea is to consider two n-sphere of fixed volume V connected by a small cylinder C of length $2L$ and radius ϵ.

The first nonzero eigenvalue converges to 0 as the radius of the cylinder goes to 0.

- We choose a function f with value 1 on the first sphere, -1 on the second, and decreasing linearly, so that the norm of its gradient is $\frac{1}{L}$.
- By construction we have $\int f dV = 0$, so that we have $\lambda_1 \leq R(f)$.
- But the Rayleigh quotient is bounded above by

$$\frac{VolC/L^2}{2V}$$

which goes to 0 as ϵ does.

Observe that we can easily fix the volume in all these constructions: so to fix the volume is no enough to have a lower bound on the spectrum.

How does the geometry allow to control the first nonzero eigenvalue in the closed eigenvalue problem?

Some classical results

- Let (M,g) be an n-dimensional compact Riemannian manifold without boundary. The Cheeger's isoperimetric constant $h = h(M)$ is defined as follows:
- $h(M) = \inf_C \{J(C);\ J(C) = \frac{Vol_{n-1}C}{min(Vol_n M_1, Vol_n M_2)}\}$, where C runs through all compact codimension one submanifolds which divide M into two disjoint connected open submanifolds M_1, M_2 with common boundary $C = \partial M_1 = \partial M_2$.

Cheeger's inequality

$$\lambda_1(M, g) \geq \frac{h^2(M, g)}{4}.$$

Buser proved thanks to a quite tricky example that Cheeger's inequality is sharp ([Bu1], thm. 1.19).

Buser's inequality

- Let (M^n, g) be a compact Riemannian manifold with Ricci curvature bounded below $Ric(M, g) \geq -\delta^2(n-1)$, $\delta \geq 0$. Then we have

$$\lambda_1(M, g) \leq C(\delta h + h^2),$$

where C is a constant depending only on the dimension and h is the Cheeger's constant.

- So we have

$$\frac{h^2}{4} \leq \lambda_1(M, g) \leq C(\delta h + h^2).$$

We cannot avoid the condition about the Ricci curvature: example at the end of the talk.

Theorem of Cheng.

Recall the fundamental question:

Try to find constants a_k and b_k depending on geometrical invariants such that, given a compact Riemannian manifold (M, g), we have

$$a_k(g) \leq \lambda_k(M, g) \leq b_k(g).$$

- Cheng Comparison Theorem. Let (M^n, g) be a compact n-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M, g) \geq (n-1)K$ and that d denote the diameter of (M, g).
- Then

$$\lambda_k(M, g) \leq \frac{(n-1)^2 K^2}{4} + \frac{C(n)k}{d^2}$$

where $C(n)$ is a constant depending only on the dimension.

This paper [Che] of Cheng is really an important reference, see MathSciNet. In particular, if $Ricci(M, g) \geq 0$, there are a lot of results in order to find the best estimate, at least for λ_1, but this is not our purpose in this introduction.

- Theorem of Li and Yau. Let (M, g) be a compact n-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M, g) \geq (n-1)K$ and that d denote the diameter of (M, g).
- Then, if $K < 0$,

$$\lambda_1(M, g) \geq \frac{exp - (1 + (1 - 4(n-1)^2 d^2 K)^{1/2})}{2(n-1)^2 d^2},$$

- If, for example, $K = -1$, we have

$$\lambda_1(M, g) \geq \frac{exp - (1 + (1 + 4(n-1)^2 d^2)^{1/2})}{2(n-1)^2 d^2},$$

- Essentially, for large d,

$$\lambda_1(M,g) \geq \frac{e^{-(2(n-1)d)}}{2(n-1)^2 d^2}.$$

- If $K=0$, then

$$\lambda_1(M,g) \geq \frac{\pi^2}{4d^2}.$$

- Why do we need to control the curvature in the inequality of Buser?

  Let (M^n, g) be a compact Riemannian manifold with Ricci curvature bounded below $Ric(M, g) \geq -\delta^2(n-1)$, $\delta \geq 0$. Then we have

  $$\lambda_1(M, g) \leq C(\delta h + h^2),$$

  where C is a constant depending only on the dimension and h is the Cheeger's constant.
- We will construct an example with h small but λ_1 not small.

- We consider a torus $S^1 \times S^1$ with its product metric g and coordinates (x, y), $-\pi \leq x, y \leq \pi$ and a conformal metric $g_\epsilon = \chi_\epsilon^2 g$.
- The function χ_ϵ is an even function depending only on x, takes the value ϵ at $0, \pi$, 1 outside an ϵ-neighbourhood of 0 and π.
- We see immediatly that the Cheeger constant $h(g_\epsilon) \to 0$ as $\epsilon \to 0$.

- It remains to see that $\lambda_1(g_\epsilon)$ is uniformly bounded from below.
- Let f be an eigenfunction for $\lambda_1(g_\epsilon)$. We have

$$R(f) = \frac{\int |df|_\epsilon^2 dV_\epsilon}{\int f^2 dV_\epsilon}.$$

- Let $S_1 = \{p : f(p) \geq 0\}$ and $S_2 = \{p : f(p) \leq 0\}$ and let $F = f$ on S_1 and $F = af$ on S_2 where a is choosen such that $\int FdV = 0$.

- This implies $R_g(F) \geq \lambda_1(g)$
- Recall that: for a compact Riemannian manifold without boundary, we have the classical fact

$$\lambda_1(M,g) = inf\{R(u) : u \neq 0; \int_M udV = 0\}.$$

- So, it is enough to show that

$$R_g(F) \leq \lambda_1(g_\epsilon).$$

- But

$$R(F) = \frac{\int_{S_1} |df|^2 dV + a^2 \int_{S_2} |df|^2 dV}{\int_{S_1} f^2 dV + a^2 \int_{S_2} f^2 dV},$$

- and

$$\int_{S_i} |df|^2 dV_\epsilon = \int_{S_i} |df|^2 dV,$$

- $$\int_{S_i} f^2 dV_\epsilon \leq \int_{S_i} f^2 dV.$$

  This implies

- $$\lambda_1(g_\epsilon) = R_{g_\epsilon}(f) \geq R(F) \geq \lambda_1(g).$$

Lecture 2: The case of the negatively curved compact manifolds

In this lecture, I will explain how the fact of being negatively curved influences the spectrum of a manifold. I first give some general results and then I will prove one of them in detail.

Most of the results are true for variable negative curvature and manifolds of finite volume. In order to avoid some technical difficulties, I will only deal with the case of compact hyperbolic manifolds, that is Riemannian manifolds with constant sectional curvature -1.

For more generality, the reader may look at [BCD].

There will be two parts:

- First, some fact of geometry that I will describe without proof (and the proofs are in general not easy).
- Then in the second part, we will see some consequences for the spectrum.

The geometry:

- First, except in dimension 2, it is difficult to construct explicitely hyperbolic manifolds.
- Most of the construction are of algebraic nature, and it is not easy to "visualize" these manifolds. However, there are some general results which allow to have a good general idea of the situation.
- A general reference for hyperbolic manifolds is the book of Benedetti and Petronio [BP].

The thick-thin decomposition. Attached to hyperbolic manifold is the so called Margulis constant $c_n > 0$ depending only on the dimension.

Define

$$M_{thin} = \{p \in M : inj(p) < c_n\},$$

where *inj* denotes the injectivity radius, and

$$M_{thick} = \{p \in M : inj(p) \geq c_n\}.$$

The main consequences of the *Margulis lemma* (see [BP],[Bu1]) are the following:

- $M_{thick} \neq \emptyset$.
- Moreover, if $n \geq 3$, M_{thick} is connected.

- M_{thin} may be empty, but if not, each connected component of M_{thin} is a tubular neighborhood of a simple closed geodesic γ of length $< c_n$.
- If $R(\gamma)$ denotes the distance between γ and M_{thick}, then

$$V(c_n/2) \leq C_n l(\gamma) \sinh R(\gamma) \leq Vol(M),$$

where $V(c_n/2)$ denote the volume of a ball of radius $c_n/2$ in the hyperbolic space, and C_n is a positive constant depending only on the dimension.

In particular, if the length of γ is small, then $R(\gamma)$ is large, of the order of $\ln(1/l(\gamma))$.

The number of connected component of M_{thin} is finite.

The structure of the volume.

- The possible values of the volume of an hyperbolic manifold is rather special (see [G]) and there is a parallel between this structure of the set of possible volume and the spectrum.
- In dimension 2, thanks to the theorem of Gauss-Bonnet, the volume of an hyperbolic surface of genus γ is $4\pi(\gamma - 1)$. But, for each genus, there is a continuous family of hyperbolic surfaces (indeed a family with $6\gamma - 6$ generators).

In dimension $n \geq 4$, given a positive number V_0, there exist only a finite number of hyperbolic n-dimensional manifolds of volume $\leq V_0$.

The case of dimension 3 is special: the set of volume admits accumulation points. They correspond to a family of 3-dimensional hyperbolic manifolds of volume $< V$ which degenerate in some sense to a non compact, finite volume hyperbolic manifold of volume V. These examples are the famous examples of Thurston, see [BP].

Implications for the spectrum

Case of surfaces, see [Bu1],[Bu4],:

- We consider the space T_γ of hyperbolic surfaces of genus γ. Then
- For each $\epsilon > 0$, there exist a surface $S \in T_\gamma$ with $\lambda_{2\gamma-3} < \epsilon$.
- This result is easy to establish: it is like construction of k small eigenvalue with the Cheeger Dumbbell.

- It was known since a long time that $\lambda_{4\gamma-3} > \frac{1}{4}$ for each $S \in T_\gamma$ and conjectured that $\lambda_{2\gamma-2} > \frac{1}{4}$ for each $S \in T_\gamma$.
- After some little progress, this conjecture was solved very recently by Otal and Rosas, see [OR].

For each $\epsilon > 0$ and each integeer $N > 0$, there exists a surface $S \in T_\gamma$ with $\lambda_N(S) \leq \frac{1}{4} + \epsilon$. This is a direct consequence of the Theorem of Cheng and of the fact that there exist surfaces with arbitrarily large diameter:

$$\lambda_N(S) \leq \frac{1}{4} + \frac{C_2 N}{d^2}.$$

Recall: Cheng Comparison Theorem. Let (M^n, g) be a compact n-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M, g) \geq (n-1)K$ and that d denote the diameter of (M, g).

Then

$$\lambda_k(M, g) \leq \frac{(n-1)^2 K^2}{4} + \frac{C(n)k}{d^2}$$

where $C(n)$ is a constant depending only on the dimension.

Case of dimension $n \geq 3$:

- The new fact is that λ_1 may be small only in the case where the volume becomes large!
- Theorem of Schoen

  There exists a constant $C(n) > 0$ such that for each compact hyperbolic manifold (M, g) of dimension $n \geq 3$ we have

$$\lambda_1(M, g) \geq \frac{C(n)}{Vol(M, g)^2}.$$

- There is however a difference between the dimension 3 and the higher dimensions.
- In dimension $n \geq 4$: Buser proved in [Bu1] that there exist a constant $C_n > 0$ such that if (M, g) is a compact hyperbolic manifold of dimension $n \geq 4$, the number of eigenvalues in the interval $[0, x]$ is bounded from above by $C_n Vol(M) x^{n/2}$ (for x large enough).

In dimension 3, it is possible to produce an hyperbolic manifold with volume bounded from above by a given constant V_0 with an arbitrarily large number of eigenvalues less than $1 + \epsilon$. This comes from the fact that the above mentionned Thurston examples have arbitrarily large diameter and volume bounded from above, and from the theorem of Cheng.

Proof of the theorem of Schoen.

Theorem of Schoen

There exists a constant $C(n) > 0$ such that for each compact hyperbolic manifold (M, g) of dimension $n \geq 3$ we have

$$\lambda_1(M, g) \geq \frac{C(n)}{Vol(M, g)^2}.$$

- Proof of Dodziuk-Randol of this theorem, see the paper [DR].
- It consists in looking at what can occur on the different parts M_{thin} and M_{thick}. The connected components of M_{thin} are simple enough to allow to do some calculations in Fermi coordinates, and to get good estimates. At the contrary, M_{thick} is complicated, but at each point the injectivity radius is large enough. This has two implications:

- we can compare the volume and the diameter: the diameter cannot be much larger than the volume, because around each point there is enough volume.
- we can use a Sobolev inequality and show that an eigenvalue associated to a very small eigenvalue is almost constant in the thick part, which is intuilively clear, but in general not true if we cannot control the injectivity radius and the curvature.
- Putting all informations together, we can prove the theorem.

- **Eigenvalues of a thin part T of M.** Recall that the thin part is a tubular neighborhood of a simple closed geodesic γ. We can endow it with the Fermi coordinates.
- A point $x = (t, \rho, \sigma) \in T$ is specified by its position t an γ, its distance ρ from γ and a point $\sigma \in S^{n-2}$. In these coordinates, the metric has the form

$$g(x) = d\rho^2 + \cosh^2 \rho dt^2 + \sinh^2 \rho d\sigma^2,$$

- the volume element is $(\sinh^{n-2} \rho \cosh \rho) d\rho dt d\sigma$.

- Let $f \neq 0$ be a function which vanishes on the boundary of T, and let us estimate its Rayleigh quotient on T.
- We will show that

$$\int_T |df|^2 \geq \frac{(n-1)^2}{4} \int_T f^2,$$

that is

$$R(f) \geq \frac{(n-1)^2}{4}.$$

- First

$$(\int_T f^2)^2 =$$

$$= (\int_{S^{n-2}} d\sigma \int_0^l dt \int_0^R f^2 (\sinh^{n-2}\rho \cosh\rho) d\rho)^2,$$

where l is the length of γ and R the radius of T (depending on t and on σ).

- We integrate by part with respect to ρ and get

$$\int_0^R f^2(\sinh^{n-2}\rho\cosh\rho)d\rho =$$

$$= -\frac{2}{n-1}\int_0^R ff_\rho \sinh^{n-1}\rho d\rho.$$

- As $\sinh\rho < \cosh\rho$, we get

- $$(\int_T f^2)^2 \leq$$
$$\frac{4}{(n-1)^2}(\int_{S^{n-2}} d\sigma \int_0^l dt \int_0^R |ff_\rho|(\sinh^{n-2}\rho \cosh\rho) d\rho)^2$$
$$= (\frac{2}{n-1})^2 (\int_T |ff_\rho|)^2.$$

- Now, by Cauchy-Schwarz inequality,

$$(\int_T |ff_\rho|)^2 \leq \int_T f^2 \int_T f_\rho^2,$$

and $f_\rho^2 \leq |df|^2$, so that we get

- $$(\int_T f^2)^2 \leq \frac{4}{(n-1)^2} \int_T f^2 \int_T f_\rho^2 \leq \int_T f^2 \int_T |df|^2,$$

and

- $$\int_T |df|^2 \geq \frac{(n-1)^2}{4} \int_T f^2.$$

Remember: Example: Monotonicity in the Dirichlet problem.

- Let $\Omega_1 \subset \Omega_2 \subset (M, g)$, two domains of the same dimension n of a Riemannian manifold (M, g). Let us suppose that Ω_1 and Ω_2 are both compact connected manifolds with boundary.
- If we consider the Dirichlet eigenvalue problem for Ω_1 and Ω_2 with the induced metric, then for each k
$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$$
with equality if and only if $\Omega_1 = \Omega_2$.

- At this stage, note that if ϕ is an eigenfunction for $\lambda_1(M)$, and if it turns out that ϕ is of constant sign on the thick part, it has to change of sign on at least one of the connected components of the thin part of M.
- This allow to construct a test function for the Dirichlet problem on a tube T with Rayleigh quotient $\lambda_1(M)$, so that we deduce that $\lambda_1(M) \geq \frac{(n-1)^2}{4}$
- this is certainly $\geq \frac{C_n}{Vol(M,g)^2}$, for a convenient constant $C(n)$, because we know that the volume of M is not arbitrarily small.

Of course, things are in general not so easy, and we have to look at the thick part of M.

- **The situation on the thick part.** In each point x of M_{thick}, the injectivity radius is at least equal to the Margulis constant $c(n)$, so that a ball of a fixed radius $r < c_n$ will be embedded. Let us denote such a ball by B.
- We want to show: On M_{thick}, there is a constant C depending only on the dimension such that

$$|\phi(y) - \phi(x)| \leq C\sqrt{\lambda_1(M)}Vol(M)^{1/2}. \qquad (2)$$

- On B, by a classical Sobolev inequality (see for example [W], 6.29, p.240), if ϕ is an eigenfunction for $\lambda_1(M)$, we have

$$|d\phi(x)| \leq C \sum_{i=0}^{N} \|\Delta^i d\phi\|_{L^2(B)},$$

  where C depend on r and on the geometry and $N = [\frac{n}{4}] + 1$.
- But we fix r and the local geometry does not change from one point to another, because of the constant curvature.

As Δ and d commute, and because ϕ is an eigenfunction, we deduce

$$|d\phi(x)| \leq C\|d\phi\|_{L^2(B)}, \tag{3}$$

and this is true for each point $x \in M_{thick}$.

- If $x, y \in M_{thick}$, we can join them by a (locally) geodesic path γ of length $\leq C_1 V$
- (the diameter of M_{thick} cannot be too large in comparison of the total volume of M)
- We choose k points $x = x_0, ..., x_k = y$ along γ such that $\gamma \subset \cup_{i=0}^{k} B(x_i, r/2)$, and such that one of these balls intersects at most β other.

Then

$$|\phi(y) - \phi(x)| \leq \sum_{i=0}^{k-1} |\phi(x_{i+1}) - \phi(x_i)| \leq$$

$$\leq C \sum_{i=0}^{k-1} \|d\phi\|_{L^2(B(x_i,r)} \leq$$

$$Ck^{1/2}(\sum_{i=0}^{k-1} \|d\phi\|^2_{L^2(B(x_i,r))})^{1/2} \leq C\beta^{1/2}k^{1/2}\|d\phi\|_{L^2(M)}$$

- Again, k is, up to a constant, at most of the order of the diameter of M_{thick}, that is of V, so that we can summarize the situation by:
- On M_{thick}, there is a constant C depending only on the dimension such that

$$|\phi(y) - \phi(x)| \leq C\sqrt{\lambda_1(M)}Vol(M)^{1/2}. \qquad (4)$$

Conclusion of the proof. We want to show

$$\lambda_1(M) \geq \frac{C(n)}{Vol(M)^2}.$$

We suppose

$$\lambda_1(M) = \frac{\epsilon}{Vol(M)^2},$$

and show that his leads to a contradiction if ϵ is too small.

- We consider an eigenfunction ϕ with $\|\phi\| = 1$.
- For $x, y \in M_{thick}$, we have $|\phi(x) - \phi(y)| < \alpha := C\frac{\epsilon^{1/2}}{Vol(M)^{1/2}}$.
- Suppose first that

$$sup\{|\phi(x)| : x \in M_{thick}\} \geq \alpha.$$

- Then things are easy, because ϕ cannot change of sign in M_{thick}. We have $\lambda_1(M) \geq \frac{(n-1)^2}{4}$.

- So we can now suppose that

$$sup\{|\phi(x)| : x \in M_{thick}\} \ < \alpha.$$

  We introduce

- $$A = \{x \in M : \phi(x) \geq \alpha\};$$

- $$B = \{x \in M : \phi(x) \leq -\alpha\};$$

- $$C = \{x \in M : |\phi(x)| < \alpha\}.$$

- We know that $A, B \subset M_{thick}$.

- Let $\phi^+ = \phi + \alpha$ and $\phi^- = \phi - \alpha$.
- ϕ^+ and ϕ^- are equal to 0 respectively on ∂B and ∂A, and this implies
- $$\int_B |d\phi|^2 = \int_B |d\phi^+|^2 \geq \frac{(n-1)^2}{4} \int_B (\phi^+)^2;$$
- $$\int_A |d\phi|^2 = \int_A |d\phi^-|^2 \geq \frac{(n-1)^2}{4} \int_A (\phi^-)^2;$$

- So

$$\int_M |d\phi|^2 \geq \int_{A\cup B} |d\phi|^2 \geq$$

$$\geq \frac{(n-1)^2}{4}\int_B (\phi^+)^2 + \frac{(n-1)^2}{4}\int_A (\phi^-)^2.$$

- But, as $\epsilon \to 0$, $|\phi - \phi^+|, |\phi - \phi^-| \to 0$ and $\int_C \phi^2 \to 0$, so that we can conclude.

Lecture 3: Estimates on the conformal class

- Let M be any compact manifold of dimension $n \geq 3$ and $\lambda > 0$.
- Then there exist a Riemannian metric g on M with $Vol(M, g) = 1$ and $\lambda_1(M, g) \geq \lambda$.

We have to add some additive constraints if we want to get upper bounds on λ_k

- It turns out that if we stay on the conformal class of a given Riemannian metric g_0, then, we get upper bounds for the spectrum on volume 1 metrics, and it is the goal of this lecture to explain this.
- Recall that the conformal class of a Riemannian metric g_0 consists of all Riemannian metric of the type $g(x) = h^2(x)g_0(x)$, where $h > 0$ is a smooth function on M.

Theorem of Korevaar.

- Let (M^n, g_0) be a compact Riemannian manifold. Then, there exist a constant $C(g_0)$ depending on g_0 such that for any Riemannian metric $g \in [g_0]$, where $[g_0]$ denotes the conformal class of g_0, then we have

$$\lambda_k(M, g) Vol(M, g)^{2/n} \leq C(g_0) k^{2/n}.$$

- Moreover, if the Ricci curvature of g_0 is nonegative, we can replace the constant $C(g_0)$ by a constant depending only on the dimension n.

- Recall that $\lambda_k(M,g)Vol(M,g)^{2/n}$ is invariant through homothety of the metric, and this control is equivalent of fixing the volume.
- The estimate is compatible with the Weyl law

$$\lambda_k(M,g) \sim \frac{(2\pi)^2}{\omega_n^{2/n}}\left(\frac{k}{Vol(M,g)}\right)^{2/n}$$

which may be also written

$$\lambda_k(M,g)Vol(M,g)^{2/n} \sim \frac{(2\pi)^2}{\omega_n^{2/n}}k^{2/n}$$

- These estimates are not sharp in general.

In the special case of surfaces, we have a bound depending only on the topology.

Let S be an oriented surface of genus γ. Then, there exist a universal constant C such that for any Riemannian metric g on S

$$\lambda_k(S, g) Vol(S) \leq C(\gamma + 1)k.$$

These results were already known for $k = 1$, with different kind of proofs and different authors (see for example the introduction of [CE1]). However, in order to make a proof for all k, Korevaar used a completely new approach.

The way to get upper bounds is to construct test functions, and it is nice to have disjointly supported functions, because, in this case, recall that

$$\lambda_k \leq sup\{R(u) : u \neq 0,\ u \in V_k\} =$$

$$= sup\{R(f_i) : i = 1, ..., k+1\}.$$

- A classical way to do this (see [Bu2], [LY]) is to construct a family of (k+1) balls of center x_i $i = 1, ..., k+1$, and radius $r = (\frac{Vol(M,g)}{Ck})^{1/n}$ such that $B(x_i, 2r) \cap B(x_j, 2r) = \emptyset$. Here $C > 0$ is a constant depending on the dimension.
- Then, construct the test function f_i with value 1 on $B(x_i, r)$, 0 outside $B(x_i, 2r)$, and for $p \in B(x_i, 2r) - B(x_i, r)$, $f_i(p) = 1 - \frac{1}{r} d(p, B(x_i, r))$.
- Then $|gradf_i(p)| \leq \frac{1}{r}$, and we have

- $$R(f_i) = \frac{\int_M |df_i|^2}{\int_M f_i^2} \leq \frac{1}{r^2} \frac{VolB(x_i, 2r)}{VolB(x_i, r)},$$

  that is

- $$R(f_i) \leq \left(\frac{k}{Vol(M, g)}\right)^{2/n} C^{2/n} \frac{VolB(x_i, 2r)}{VolB(x_i, r)}.$$

- So, we see that we need to control the ratio

$$\frac{VolB(x_i, 2r)}{VolB(x_i, r)}.$$

- This depend a lot of what we know on the Ricci curvature.

- Namely, we have the Bishop-Gromov inequality: if $Ricci(M,g) \geq -(n-1)a^2 g$, with $a \geq 0$, then for $x \in M$ and $0 < r < R$,

$$\frac{VolB(x,R)}{VolB(x,r)} \leq \frac{VolB^a(x,R)}{VolB^a(x,r)}$$

where B^a denote the ball on the model space of constant curvature $-a^2$.

- So, if $a > 0$, the control of the ratio $\frac{VolB(x_i, 2r)}{VolB(x_i, r)}$ is exponential in r and becomes bad for large r.
- If $a = 0$, that is if $Ricci(M, g) \geq 0$, the ratio $\frac{VolB(x_i, 2r)}{VolB(x_i, r)}$ is controled by a similar ratio but in the Euclidean space, and this depend only on the dimension!

However, when we look in a conformal class of a given Riemannian metric g_0, we have a priori no control on the curvature, so it seems hopeless to get such test functions. This is precisely the contribution of N. Korevaar to develop a method which allows to deal with such situations.

The construction of Grigor'yan-Netrusov-Yau.

The construction is a rather metric construction so that we can present it on the context of metric measured spaces.

Let (X, d) be a metric space. The *annuli*, denoted by $A(a, r, R)$, (with $a \in X$ and $0 \leq r < R$) is the set

$$A(a, r, R) = \{x \in X : r \leq d(x, a) \leq R\}.$$

Moreover, if $\lambda \geq 1$, we will denote by λA the annuli $A(a, \frac{r}{\lambda}, \lambda R)$.

- Let a metric space (X, d) with a finite measure ν. We make the following hypothesis about this space:
- The ball are precompact (the closed balls are compact);
- The measure ν is non atomic;
- There exist $N > 0$ such that, for each r, a ball of radius r may be covered by at most N balls of radius $r/2$.
- This hypothesis plays, in some sens, the role of a control of the curvature, but, as we will see, it is much weaker. Note that it is purely metric, and has nothing to do with the measure.

- If these hypothesis are satisfied, we have the following result
- For each positive integer k, there exist a family of annuli $\{A_i\}_{i=1}^k$ such that
- We have $\nu(A_i) \geq C(N)\frac{\nu(X)}{k}$, where $C(N)$ is a constant depending only on N;
- The annuli $2A_i$ are disjoint from each other.

Proof of Theorem of Korevaar

- Let (M^n, g_0) be a compact Riemannian manifold. Then, there exist a constant $C(g_0)$ depending on g_0 such that for any Riemannian metric $g \in [g_0]$, where $[g_0]$ denotes the conformal class of g_0, then we have

$$\lambda_k(M, g) Vol(M, g)^{2/n} \leq C(g_0) k^{2/n}.$$

- Moreover, if the Ricci curvature of g_0 is nonegative, we can replace the constant $C(g_0)$ by a constant depending only on the dimension n.

- The metric space X will be the manifold M with the Riemannian distance associated to g_0 (and which has nothing to do with g).
- The *measure* ν will be the measure associated to the volume form dV_g.
- As M is compact, the theorem of Bishop-Gromov give us a constant $C_1(g_0)$ such that, for each $r > 0$ and $x \in M$,

$$\frac{Vol_{g_0} B(x,r)}{Vol_{g_0} B(x,r/2)} \leq C_1(g_0).$$

- We know that $C_1(g_0)$ will depend on the lower bound of $Ricci(g_0)$ and of the diameter of (M, g_0).

- As the distance depends only on g_0 we have a control on the number of ball of radius $r/2$ we need to cover a ball of radius r, thanks to a classical packing lemma, see [Zu].
- Also, there exist $C_2 = C_2(g_0)$ such that, for all $r \geq 0$ and $x \in M$,

$$Vol_{g_0}(B(x,r)) \leq C_2 r^n.$$

In general, these constant are bad: we can only say, and this is the point for our theorem, that they depend only on g_0 and not on g. But if $Ricci(g_0) \geq 0$, then the Bishop-Gromov theorem allows us to compare with the euclidean space, and these constants depend only on the dimension !

- In order to estimate $\lambda_k(g)$, we use a family of $2k+2$ annuli given by the construction of Grigor'yan-Netrusov-Yau and satisfying $Vol_g(A_i) \geq C_3(g_0)\frac{Vol_g(M)}{k}$. Here, the constant C_3 depends on g_0 via $C_1(g_0)$,as indicated in [GNY].
- As the annuli $2A_i$ are disjoint, we use them to construct test functions with disjoint support.

- For an annuli $A(a, r, R)$ we will consider a function taking the value 1 in A, 0 outside $2A$, and decreasing proportionaly to the distance between A and $2A$. Let us estimate the Rayleigh quotient

$$R(f) = \frac{\int_{2A} |df|_g^2 dV_g}{\int_{2A} f^2 dV_g}$$

of such a function.

- We have, thanks to an Holder inequality,

$$\int_{2A} |df|_g^2 dV_g \leq (\int_{2A} |df|_g^n dV_g)^{2/n} Vol_g(2A))^{1-2/n}.$$

- Recall that, if $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_{2A} uv \ dV_g \leq (\int_{2A} u^p dV_g)^{1/p} (\int_{2A} v^q dV_g)^{1/q}$$

and we use this for $p = n/2, q = \frac{n}{n-2}$, $u = |df|_g^2$, $v = 1$.

- At this stage, we use the conformal invariance:

$$(\int_{2A} |df|_g^n dV_g)^{2/n} = (\int_{2A} |df|_{g_0}^n dV_{g_0})^{2/n},$$

- The conformal invariance comes from the fact that

$$|df|_g^n = |df|_{h^2 g_0}^n = \frac{1}{h^n}|df|_{g_0}^n$$

and

$$dV_g = dV_{h^2 g_0} = h^n dV_{g_0}.$$

- Because
$$|df|_{g_0} \leq \frac{2}{r}$$
(resp. $\frac{2}{R}$) and
- $$Vol_{g_0}(B(x,r)) \leq C_2(g_0) r^n$$
we have
- $$\left(\int_{2A} |df|_g^n dV_g\right)^{2/n} = \left(\int_{2A} |df|_{g_0}^n dV_{g_0}\right)^{2/n} \leq C_2(g_0) 2^n.$$

- Moreover, we know that

$$Vol_g(A) \geq C_3(g_0)\frac{Vol_g(M)}{k}.$$

- As we have $2k+2$ annuli, at least $k+1$ of them have a measure less than $\frac{Vol_g(M)}{k}$.
- So,

$$R(f) \leq \frac{(C_2(g_0)2^n)^{2/n}Vol_g(M)^{(n-2)/n}k}{C_3(g_0)k^{(n-2)/n}Vol_g(M)} =$$

$$= C(g_0)(\frac{k}{Vol_g(M)})^{2/n}.$$

If $Ric_{g_0} \geq 0$, the constants C_1 and C_2 depend only on n, and the same is true for C_3, and so, also for C.

Futher applications

When we know that we have upper bounds, we can investigate things from a quantitative or qualitative viewpoint. Let us give without any proof the example of the *conformal spectrum* and of the *topological spectrum* we developped with El Soufi.

- For any natural integer k and any conformal class of metrics $[g_0]$ on M, we define the conformal k-th eigenvalue of $(M, [g_0])$ to be
-

$$\lambda_k^c(M, [g_0]) =$$
$$= \sup \left\{ \lambda_k(M, g) Vol(M, g)^{2/n} : \; g \in [g_0] \right\}.$$

- The sequence $\{\lambda_k^c(M, [g_0])\}$ constitutes the conformal spectrum of $(M, [g_0])$.

In dimension 2, one can also define a topological spectrum by setting, for any genus γ and any integer $k \geq 0$,

$$\lambda_k^{top}(\gamma) = \sup \left\{ \lambda_k(M,g) Vol(M,g) \right\},$$

where g describes the set of Riemannian metric on the orientable compact surface M of genus γ.

There are some difficult questions about the conformal spectrum:

- Is the supremum a maximum, that it does it exist a Riemannian metric $g \in [g_0]$ where $\lambda_k Vol(M, g)^{2/n}$ is maximum?
- It is hopeless to determine $\lambda_k[g_0]$ in general, but shall we say something in the case of the sphere, for example?

We have the following qualitative theorem:

For any conformal class $[g]$ on M and any integer $k \geq 0$,

$$\lambda_k^c(M,[g]) \geq \lambda_k^c(\mathbb{S}^n,[g_s]),$$

where g_s is the canonical metric on the sphere $\mathbb{S}^n$.

For any conformal class $[g]$ on M and any integer $k \geq 0$,

$$\lambda_{k+1}^c(M,[g])^{n/2} - \lambda_k^c(M,[g])^{n/2} \geq \lambda_1^c(\mathbb{S}^n,[g_s]) = n^{n/2}\omega_n,$$

where ω_n is the volume of the n-dimensional Euclidean sphere of radius one.

For any conformal class $[g]$ on M and any integer $k \geq 0$,

$$\lambda_k^c(M,[g]) \geq n\omega_n^{2/n} k^{2/n}.$$

This implies

$$n\omega_n^{2/n}k^{2/n} \leq \lambda_k^c(M,[g]) \leq Ck^{2/n}$$

for some constant C (depending only on n and a lower bound of $Ric\ d^2$, where Ric is the Ricci curvature and d is the diameter of g or of another representative of $[g]$).

- Corollary implies also that, if the k-th eigenvalue $\lambda_k(g)$ of a metric g is less than $n\omega_n^{2/n}k^{2/n}$, then g does not maximize λ_k on its conformal class $[g]$.
- For instance, *the standard metric g_s of $\mathbb{S}^2$, which maximizes λ_1, does not maximize λ_k on $[g_s]$ for any $k \geq 2$.* This fact answers a question of Yau (see [Y], p. 686).

Lecture 4: The spectrum of submanifolds of the euclidean space

- In this lecture, we will consider submanifolds of the euclidean space. Some of the results I will give may be generalized for other spaces, for example the hyperbolic space, and this is more or less difficult depending on the question
- The goal of this lecture is also to present a classical method to get upper bound, namely the use of coordinates functions. But it applies for λ_1 and in general not for the other eigenvalues.

Two typical results for the first nonzero eigenvalue:

- Theorem of Reilly (1977): Let M^m be a compact submanifold of dimension m of $\mathbb{R}^n$. Then,

$$\lambda_1(M) \leq \frac{m}{\mathrm{Vol}(M)} \|H(M)\|_2^2 = \frac{m}{\mathrm{Vol}(M)} \int_M H^2 dV$$

  where $\|H(M)\|_2$ is the L^2-norm of the mean curvature vector field H of M.
- Moreover, the inequality is sharp, and the equality case correspond exactly to the case where M is isometric to a round sphere of dimension m.

This result was generalized to the submanifolds of the sphere and of the hyperbolic space by Grosjean [Gr] and to hypersurfaces of rank 1 symmetric spaces by Santhanam [San].

Theorem of Chavel (1978)

- Let Σ be an embedded compact hypersurface bounding a domain Ω in $\mathbb{R}^{n+1}$. Then
-

$$\lambda_1(\Sigma) Vol(\Sigma)^{2/n} \leq \frac{n}{(n+1)^2} I(\Omega)^{2+\frac{2}{n}}, \qquad (5)$$

where $I(\Omega)$ is the isoperimetric ratio of Ω,

- that is

$$I(\Omega) = \frac{Vol(\Sigma)}{Vol(\Omega)^{n/(n+1)}}.$$

Moreover, equality holds in (6) if and only if Σ is embedded as a round sphere.

Indeed, Chavel proved this theorem for hypersurface of a Cartan-Hadamard manifold (complete, simply connected manifold, with non positive sectional curvature).

- These results lead to natural questions
- Question 1: is it possible to generalize these results to other eigenvalues.
- Question 2: Is it really necessary to impose conditions on the curvature or on the isoperimetric ratio, at least for hypersurfaces?

The answer to the second question is yes: namely, in [CDE], we show that, for $n \geq 2$, it is possible to produce an hypersurface of $\mathbb{R}^{n+1}$ with volume 1 and arbitrarily large first nonzero eigenvalue. If $n \geq 3$, we can even prescribe the topology.

The answer to the first question is also yes, but the generalization is not easy.

Proof of Theorem of Chavel

Let Σ be an embedded compact hypersurface bounding a domain Ω in $\mathbb{R}^{n+1}$. Then

$$\lambda_1(\Sigma) Vol(\Sigma)^{2/n} \leq \frac{n}{(n+1)^2} I(\Omega)^{2+\frac{2}{n}}, \qquad (6)$$

where $I(\Omega)$ is the isoperimetric ratio of Ω.

Moreover, equality holds in (6) if and only if Σ is embedded as a round sphere.

- We will present the proof of the Theorem of Chavel by using a very classical method coming from Hersch: the use of coordinates functions (we speak sometimes from *barycentric* methods).
- The idea is to use the restriction to Σ of the coordinates functions of $\mathbb{R}^{n+1}$as test functions.

- If we have

$$a_i = \int_\Sigma x_i dV_\Sigma,$$

then

- $$\int_\Sigma (x_i - \frac{a_i}{Vol(\Sigma)}) dV_\Sigma = 0.$$

- So a change of coordinates (or by putting the origine at the barycenter of Σ), we can suppose

$$\int_{\Sigma} x_i dV_{\Sigma} = 0$$

for $i = 1, ..., n+1$ This mean that we have in the hands $(n+1)$ test functions in order to find an upper bound for $\lambda_1(\Sigma)$.

- We know that

$$\lambda_1(\Sigma) \leq \frac{\int_{\Sigma} |grad\ \ x_i|^2_{\Sigma} dV_{\Sigma}}{\int_{\Sigma} x_i^2 dV_{\Sigma}}$$

for $i = 1, ..., n+1$.

The goal is now to rely the quantities

$$\int_\Sigma |grad\ x_i|_\Sigma^2 dV_\Sigma \text{ and } \int_\Sigma x_i^2 dV_\Sigma$$

to the isoperimetric ratio, that is to quantities like $Vol(\Sigma)$ and $Vol(\Omega)$.

- We introduce the position vector field X on $\mathbb{R}^{n+1}$, given by $X(x) = x$.
- We get immediatly $div\ X = n + 1$.
- The Green formula says that

$$\int_\Omega div\ X dV_\Omega = \int_\Sigma \langle X, \nu \rangle dV_\Sigma,$$

where ν is the outward normal vector field of Σ with respect to Ω.

- This implies

$$(n+1)Vol(\Omega) \leq \int_{\Sigma} |X| dV_{\Sigma} \leq$$

$$\leq Vol(\Sigma)^{1/2} (\int_{\Sigma} |X|^2 dV_{\Sigma})^{1/2}$$

- So, we express $|X|^2$ and get

$$(n+1)Vol(\Omega) \leq Vol(\Sigma)^{1/2} (\int_{\Sigma} (\sum_{i=1}^{n+1} x_i^2) dV_{\Sigma})^{1/2}.$$

- At this stage we use the relation

$$\int_\Sigma |grad\ x_i|_\Sigma^2 dV_\Sigma \geq \lambda_1(\Sigma) \int_\Sigma x_i^2 dV_\Sigma.$$

- We have

$$(n+1)Vol(\Omega) \leq Vol(\Sigma)^{1/2}\Big(\int_\Sigma \big(\sum_{i=1}^{n+1} x_i^2\big) dV_\Sigma\Big)^{1/2} \leq$$

$$\leq \Big(\frac{Vol(\Sigma)}{\lambda_1(\Sigma)}\Big)^{1/2}\Big(\int_\Sigma \big(\sum_{i=1}^{n+1} |grad\ x_i|_\Sigma^2 dV_\Sigma\Big)^{1/2}.$$

- So we need to control this last term: for $x \in \Sigma$, we introduce a orthonormal basis $F_1, ..., F_n$ of $T_x\Sigma$, and note that *grad* $x_i = e_i$ in $\mathbb{R}^{n+1}$.

- On Σ, we have

$$grad\ x_i = \sum_{j=1}^{n} \langle grad\ x_i, F_j \rangle F_j,$$

so that

-

$$\sum_{i=1}^{n+1} |grad\ x_i|_{\Sigma}^2 = \sum_{i=1}^{n+1} \sum_{j=1}^{n} \langle grad\ x_i, F_j \rangle^2 =$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n+1} \langle grad\ x_i, F_j \rangle^2 = \sum_{j=1}^{n} |F_j|^2 = n.$$

- We can summarize this by

$$\lambda_1(\Sigma) \leq \frac{Vol(\Sigma)^2}{Vol(\Omega)^2} \frac{n}{(n+1)^2},$$

which is indeed the result of Chavel's paper.

- We immediatly deduce

$$\lambda_1(\Sigma) Vol(\Sigma)^{2/n} \leq \frac{n}{(n+1)^2} I(\Omega)^{2+\frac{2}{n}}.$$

- To finish the proof, we have to study the equality case: to have equality means that all inequalities become equalities. In particular, at each point $x \in \Sigma$, we have $|X| = \langle X, \nu \rangle$.
- This implies that X is proportional to ν. If we have an hypersurface such that the position vector proportional to the normal vector, this is a round sphere.

Some generalizations

If we want to generalize these results for other λ_k, it is hopeless to use the same barycentric method as for λ_1.

- Concerning results of the type Reilly, there were generalized recently by El Soufi, Harrell and Illias [EHI]:
- Let M^m be a compact submanifold of $\mathbb{R}^n$. Then, for any positive integer k,

$$\lambda_k(M) \leq R(m)\|H(M)\|_\infty^2 \, k^{2/m},$$

where $\|H(M)\|_\infty$ is the L^∞-norm of $H(M)$ and $R(m)$ is a constant depending only on m.

Concerning upper bounds in terms of the isoperimetric ratio, we have the following result in [CEG] (see El Soufi's talk in this congress):

For any bounded domain $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,

$$\lambda_k(\Sigma) Vol(\Sigma)^{2/n} \leq \gamma_n I(\Omega)^{1+2/n} k^{2/n} \tag{7}$$

with γ_n is a positive constant depending only on n.

In order to prove this theorem, the idea is again to find a good set of test functions, and, in order to find these test functions, to find a nice covering of Σ with disjoint sets. To this aim, we can use a method developped with D. Maerten in [CMa]. This method is in the spirit of what did Grigor'yan, Netrusov and Yau, and it has a lot of applications. The main result is as follow:

- Let (X, d, μ) be a complete, locally compact metric measured space, where μ is a finite measure. We assume that for all $r > 0$, there exists an integer $N(r)$ such that each ball of radius $4r$ can be covered by $N(r)$ balls of radius r.
- If there exist an integer $K > 0$ and a radius $r > 0$ such that, for each $x \in X$

$$\mu(B(x, r)) \leq \frac{\mu(X)}{4N^2(r)K},$$

Then, there exist K μ-measurable subsets $A_1, \ldots, A_K$ of X such that, $\forall i \leq K$, $\mu(A_i) \geq \frac{\mu(X)}{2N(r)K}$ and, for $i \neq j$, $d(A_i, A_j) \geq 3r$.

Some open questions

Open question 1: It is related to this lecture.

There are some results for λ_1 obtained with barycentric methods that we are (at the moment) not able to generalize to other eigenvalues. An emblematic example is a Theorem du to El Soufi and Ilias [EI2].

They consider a Riemannian manifold (M^m, g) and look at a Schroedinder operator, namely $\Delta_q = \Delta_g + q$ where Δ_g is the usual Laplacian, and q is a C^∞ potential. We also denote by $\bar{q}$ the mean of q on M, namely $\bar{q} = \frac{1}{Vol(M,g)} \int_M q dV_g$.

- Then, El Soufi and Ilias study the second eigenvalue of $\Delta + q$, denoted by $\lambda_1(\Delta_g + q)$ (and which correspond to the "usual" λ_1 when q is 0) for g on the conformal class of a given metric g_0.
- We have

$$\lambda_1(\Delta_g + q) \leq m\left(\frac{VC(g_0)}{Vol(M, g)}\right)^{2/m} + \bar{q}$$

where $VC(g_0)$ is a conformal invariant, the *conformal volume*.

- The generalization of this theorem to higher eigenvalues is not known.

- Open question 2: This question is related to the lecture 3.
- When we know that the supremum of the functional λ_k is bounded on a certain set of metric (a.e. the conformal class of a given Riemannian metric), it may be interesting to look at qualitative results in the spirit of the results obtained with El Soufi, and that I described in lecture 3.
- I give two situations where this may be interesting (and not trivial).

- **Case 1:** We consider the Neumann problem for domains Ω (bounded, smooth boundary) of the hyperbolic space $\mathbb{H}^n$.
- Let

$$\nu_k(V) = sup_{\Omega\subset\mathbb{H}^n}\{\nu_k(\Omega) : Vol(\Omega) = V\},$$

where ν_k denotes the k-th eigenvalue for the Neumann problem. It is known that this supremum exists (see for example [CMa]).

- Then it is interesting to study this spectrum:
- Is $\nu_{k+1}(V) - \nu_k(V) > 0$?
- If the answer is yes, it it possible to estimate the gap?
- How does $\nu_k(V)$ depend on V?

Note that the same question for the euclidean space is not so interesting: we can do more or less the same as we did with A. El Soufi for the conformal spectrum.

- **Case 2:** We consider the set of compact, convex embedded hypersurfaces of the euclidean space.
- Let
$$\lambda_k = sup_\Sigma\{\lambda_k(\Sigma)\},$$
where Σ describes the set of convex hypersurface of volume 1. It is known that this supremum exists (see [CDE]).
- What about $\lambda_{k+1} - \lambda_k$?
- What can be said in the special case of λ_1? We may think that the supremum is given by the round sphere.

Open question 3: A lot of questions concern the Hodge Laplacian, that is the Laplacian acting on p-form. One interesting question concerns the compact 3-dimensional hyperbolic manifolds.

- It was shown in [CC] that when a family of compact hyperbolic 3-manifolds degenerates to a non compact manifold of finite volume, it forces the apparition of small eigenvalues for 1-forms. The eigenvalues we constructed are $\leq \frac{C}{d^2}$ where C is a universal constant and d is the diameter.
- The question is to decide whether or not we have a lower bound of the type $\frac{C}{d^2}$, or if we can construct much smaller eigenvalues.

There are some partial answers in [MG], [Ja], but the question is open. One of the interest is that the topology of the manifolds of the degenerating family will certainly play a role and has to be well understood and related to the spectrum.