Spectral inequalities for Quantum Graphs

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joint work with Evans M. Harrell, II

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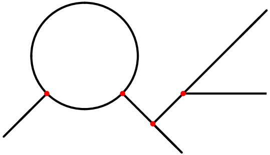
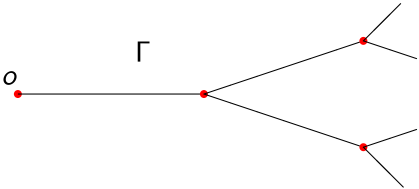
Metric Graphs, Metric Trees



- $\mathcal{V}$ . . . set of vertices
- $\mathcal{E}$ . . . set of edges, i.e. intervals



- metric defined by the distance, e.g. $|x| = \text{dist}(o, x)$

Quantum Graphs

Define Schrödinger operator $H(\alpha)$ in $L^2(\Gamma) = \bigoplus_{e\in\mathcal{E}} L^2(e)$, acting as

$$H(\alpha)\phi = -\frac{d^2\phi}{dx^2} + V(x)\phi \quad \text{on every edge } e,$$

where $\phi \in D(H(\alpha)) \quad \Leftrightarrow$

$$\phi \in H^2(e) \quad \forall\, e \in \mathcal{E}, \quad \sum_{e\in\mathcal{E}} \|\phi\|^2_{H^2(e)} < \infty$$

with Kirchhoff (Neumann) b.c. in every vertex $v \in \mathcal{V}$, i.e. ϕ is continuous in v and

$$\sum_{e\in E_v} \frac{d\phi}{dx}(v) = 0 \qquad \forall\, v \in \mathcal{V}.$$

Quantum Graphs

assumption:

$$\lim_{|x|\to\infty} V(x) = 0,$$

then

$$\sigma_{ess}(H(\alpha)) = [0, \infty)$$

and

$$\sigma_d(H(\alpha)) = \{-E_j(\alpha) < 0, \; j \in \mathbb{N}\}$$

How do the eigenvalues $-E_j(\alpha)$ depend on V?

$$\Downarrow$$

spectral inequalities: $\sum_j E_j^\gamma(\alpha) \leq \, ?$

Lieb-Thirring inequalities for metric trees

Theorem [D., Harrell]

Let Γ be a metric tree with a finite number of vertices and edges and $V \in L^{\gamma+1/2}(\Gamma)$. Then the *Lieb-Thirring inequality* (LTI)

$$\alpha^{1/2} \sum_j E_j^\gamma(\alpha) \leq L^{cl}_{\gamma,1} \int_\Gamma (V_-(x))^{\gamma+1/2} \, dx$$

holds for all $\alpha > 0$ and $\gamma \geq 2$.

$L^{cl}_{\gamma,1}$ ... *classical constant*.

Classical Lieb-Thirring inequalities in $\mathbb{R}^d$

$$H(\alpha) = -\alpha\Delta + V(x) \quad \textit{in } L^2(\mathbb{R}^d),\ \alpha > 0.$$

Lieb-Thirring inequalities

$$\alpha^{d/2} \sum_j E_j^\gamma(\alpha) \le L_{\gamma,d} \int_{\mathbb{R}^d} (V_-(x))^{\gamma+d/2}\, dx,$$

for suitable values of γ (depending on d) and

$$L_{\gamma,d} \ge L_{\gamma,d}^{cl} := \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma+1+d/2)}$$

Weyl's law:

$$\lim_{\alpha\to 0+} \alpha^{d/2} \sum E_j^\gamma(\alpha) = L_{\gamma,d}^{cl} \int_{\mathbb{R}^d} (V_-(x))^{\gamma+d/2}\, dx.$$

Stubbe's proof of sharp LT for $\gamma \geq 2$

- A trace formula ("sum rule") of Harrell-Stubbe '97, for $H = -\alpha\Delta + V$:

$$\sum E_j^\gamma(\alpha) - \alpha \frac{2\gamma}{d} \sum E_j^{\gamma-1}(\alpha) \|\nabla \phi_j\|^2 = \text{explicit expr.} \leq 0.$$

- $< \phi_j, -\Delta\phi_j >= \|\nabla \phi_j\|^2 = \frac{\partial E_j(\alpha)}{\partial \alpha}$ (Feynman-Hellman)

Thus

$$\sum E_j^2(\alpha) \leq \frac{2\alpha}{d} \frac{\partial}{\partial \alpha} \sum E_j^2(\alpha)$$

$$\frac{\partial}{\partial \alpha} \left(\alpha^{d/2} \sum E_j^2(\alpha) \right) \leq 0.$$

Stubbe's proof of sharp LTI for $\gamma \geq 2$

$$\frac{\partial}{\partial \alpha}\left(\alpha^{d/2}\sum E_j^2(\alpha)\right) \leq 0.$$

And classical LTI is an immediate consequence for $\gamma \geq 2$!

$$\alpha^{d/2}\sum E_j^{\gamma}(\alpha) \leq \lim_{\alpha\to 0+}\alpha^{d/2}\sum E_j^{\gamma}(\alpha) = L_{\gamma,d}^{cl}\int_{\mathbb{R}^d}(V_-(x))^{\gamma+d/2}\,dx.$$

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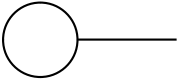
Does an analog of the LTI ($d=1$) hold also for QGraphs with the same sharp constants $L_{\gamma,1}^{cl}$?

in general: NO!



What about QGraphs?

For metric trees:

$$\sum E_j^{\gamma}(\alpha) \leq C_\gamma \int_\Gamma (V_-(x))^{\gamma+1/2}\, dx, \quad \gamma \geq 1/2 \quad \text{[E, F, K]}$$

$$C_\gamma = L^{cl}_{\gamma,1}?$$

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$$C_\gamma = L^{cl}_{\gamma,1}?$$

YES for $\gamma \ge 2$

Theorem [D., Harrell]
Let Γ be a metric tree with a finite number of vertices and edges and $V \in L^{\gamma+1/2}(\Gamma)$. Then the *Lieb-Thirring inequality* (LTI)

$$\alpha^{1/2} \sum_j E_j^\gamma(\alpha) \le L^{cl}_{\gamma,1} \int_\Gamma (V_-(x))^{\gamma+1/2}\, dx$$

holds for all $\alpha > 0$ and $\gamma \ge 2$.

Idea of proof:

- sum rule of Harrell and Stubbe:
  $\frac{1}{2}\sum E_j^2 < [G,[H,G]]\phi_j, \phi_j > -E_j\|[H,G]\phi_j\|^2 \leq 0,$

  $G\phi_j \in D(H_\Gamma)?$

- averaging argument:
  family of piecewise affine functions G
  $\Rightarrow$ reduction of the problem to a combinatorial problem
- Stubbe's monotonicity argument