# Sums of Reciprocal Eigenvalues

Bodo Dittmar
Martin Luther University Halle-Wittenberg (Germany)
E-mail: bodo.dittmar@mathematik.uni-halle.de

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**Contens**

## 1 Introduction

The eigenvalue problem of the fixed membrane

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } D, \\ u &= 0 \text{ on } \partial D, \end{aligned} \tag{1}$$

the eigenvalue problem of the free membrane

$$\begin{aligned} \Delta v + \mu v &= 0 \text{ in } D, \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \partial D \end{aligned} \tag{2}$$

$n$ stands for the normal to $\partial D$, $\lambda$ and $\mu$ for the eigenvalue parameters. It is well-known that there exists an infinity of eigenvalues with finite multiplicity

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 ...,$$

$$0 = \mu_1 < \mu_2 \leq \mu_3 ... .$$

Furthermore the method mentioned works also for the Stekloff problems

$$\begin{aligned} \Delta u &= 0 \text{ in } D, \\ \frac{\partial u}{\partial n} &= \nu u \text{ on } \partial D, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \Delta u &= 0 \text{ in } D, \\ \frac{\partial u}{\partial n} &= \sigma u \text{ on } C_1, \\ u &= 0 \text{ on } C_2, \end{aligned} \tag{4}$$

where $\partial D = C = C_1 \cup C_2$. There is also an infinity of eigenvalues with finite multiplicity

$$0 < \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq ..., \quad 0 = \nu_1 < \nu_2 \leq \nu_3 \leq ... .$$

# 2 Membrane problems

## 2.1 Fixed membrane

**Theorem 1** *(Pólya, Schiffer 1954) For any* $n$

$$\frac{1}{\dot{r}^2}\sum_{i=1}^{n}\frac{1}{\lambda_i} \geq \sum_{i=1}^{n}\frac{1}{\lambda_i^{(o)}}$$

*where* $\lambda_i^{(o)}$ *denotes the eigenvalues of the unit disk. Equality holds if and only if* $D$ *is the unit disk.*

$U = \{z : |z| < 1\}$

$$\Delta u + \lambda u|f'(z)|^2 = 0 \text{ in } U,$$
$$u|_{\partial U} = 0,$$
$$\int_U u_i u_j |f'(z)|^2 dA_z = \delta_{ij},\ i,j = 1,2,\dots ,$$

$$u_j(\zeta) = \lambda_j \int_U G(z,\zeta)|f'(z)|^2 u_j(z) dA_z.$$

$$G(z,\zeta)|f'(z)||f'(\zeta)| = \sum_{j=1}^{\infty} \frac{u_j(z)|f'(z)|u_j(\zeta)|f'(\zeta)|}{\lambda_j}$$

**Lemma 1** *For the eigenvalues of the fixed membrane problem holds*

$$\sum_1^n \frac{1}{\lambda_j} = \max_{L_{n-1}} \sum_1^n \int_U \int_U G_U(z,\zeta)|f'(z)|^2 |f'(\zeta)|^2 v_j(z) v_j(\zeta) dA_z dA_\zeta,$$

*where* $\int_U |f'(z)|^2 v_i(z) v_j(z) dA_z = \delta_{i,j}, i,j = 1,2,...,n, v_j \in L^2(U)$ *and* $v_j$ *is a basis of the space* $L_{n-1}$.

$$v_j = \sum_{i=1}^j c_{ji} u_i^{(o)}, c_{jj} \neq 0,$$

$$\sum_{k=1}^n \frac{1}{\lambda_k} \geq \sum_{k=1}^n \frac{\int_U {u_k^{(o)}}^2 |f'(z)|^2}{\lambda_k^{(o)}}$$

**Theorem 2** *Let $u_k^{(o)}$ be the eigenfunctions of the fixed membrane problem in the unit disk, $\lambda_k$ the corresponding eigenvalues and let $f(z) = z + a_2 z^2 + \ldots$ be a conformal mapping of the unit disk onto $D$. Then, for any $n \geq 2$ we have*

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \geq \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} + \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} \sum_{j=2}^{\infty} j^2 |a_j|^2 \int_U {u_k^{(o)}}^2 r^{2j-2} dA$$

**Corollary 1**

$$\sum_{k=1}^{n} \frac{1}{\lambda_k^2} \geq -\sum_{k=1}^{n} \frac{1}{{\lambda_k^{(o)}}^2} + 2 \int_U |f'(z)|^2 \int_U G_U^2(z,\zeta) dA_\zeta dA_z$$

$$\int_U G_U^2(z,\zeta) dA_z = \frac{\pi}{2} - \frac{3}{4}\pi\rho^2 + \pi \sum_{1}^{\infty} \frac{\rho^{2n+2}}{n(n+1)} - \pi \sum_{2}^{\infty} \frac{\rho^{2n}}{n^2-1}, \rho = |\zeta|$$

$$\int_U \int_U G_U^2(z,\zeta) dA_z dA_\zeta = \sum_{k=1}^{n} \frac{1}{{\lambda_k^{(o)}}^2}$$

## 2.2 Free membrane

**Lemma 2** *Let $N_f(z,\zeta)$ be the following symmetric function depending on an univalent conformal map $f$*

$$N_f(z,\zeta) = A\ N(z,\zeta) + H_f(z) + H_f(\zeta), z, \zeta \in U,$$

*where*

$$H_f(z) = -\int_U |f'(\zeta)|^2 N(z,\zeta) dA_\zeta, z \in U$$

*and $A = \int_U |f'(z)|^2 dA_z < \infty$. Then*

$$\begin{aligned} \Delta_z N_f(z,\zeta) &= |f'(z)|^2, z \neq \zeta, z \in U, \\ \frac{\partial N_f(z,\zeta)}{\partial n_z} &= 0 \text{ on } \partial U, \\ H_f(z) &= \frac{1}{4}|f(z)|^2 + h(z), \end{aligned}$$

*where $h(z)$ is a harmonic function in $U$ with $\frac{\partial h}{\partial n} = A/(2\pi) - 1/4\frac{\partial |f|^2}{\partial n}$ on $\partial U$ for a sufficiently smooth $f(z)$ and $n$ is the outward pointing normal.*
*In particular*

$$H_{f\equiv z}(z) = \frac{1}{4}|z|^2.$$

$$v_j(\zeta) = \frac{\mu_j}{A} \int_U N_f(z,\zeta) v_j(z) |f'(z)|^2 dA_z, j = 2, 3, \dots ,$$

where $A = \int_U |f'(z)|^2 dA < \infty$ and the kernel $N_f(z,\zeta)|f'(z)||f'(\zeta)|$ with the eigenfunctions $v_j(z)|f'(z)|$ with the eigenvalues $\mu_j/A$ in the space $V_f$.

**Lemma 3**

$$\sum_{j=2}^{n} \frac{A}{\mu_j} = \max_{L_{n-1}} \sum_{i=2}^{n} \int_U \int_U (N_f(z,\zeta) - AC)|f'(z)|h_i(z)|f'(\zeta)|h_i(\zeta) dA_z dA_\zeta,$$

*where* $\{h_i\}_{i=2}^n$ *is a basis of* $L_{n-1}$ *satisfying the orthonormality conditions* $\int_U h_i h_j dA = \delta_{ij}$.

**Lemma 4** *Let $v_m^{(o)}$ be the eigenfunctions of the free membrane problem in the unit disk and let $f(z) = z + a_2 z^2 + ...$ be a conformal mapping of the unit disk onto $D$. For a radial eigenfunction we have for any conformal mapping*

$$A \int_U {v_m^{(o)}}^2 |f'(z)|^2 dA - \left( \int_U v_m^{(o)} |f'(z)|^2 dA \right)^2 \geq A,$$

*where $A$ is the area of the domain $D$. Let $v_m^{(o)}(z)$ and $v_{m+1}^{(o)}(z)$ be the normalized eigenfunctions of the unit disk belonging to the same eigenvalue $\mu_m^{(o)}$, such that $(v_m^{(o)}(z))^2 + (v_{m+1}^{(o)}(z))^2$ is radial. Then*

$$A \int_U ({v_m^{(o)}}^2 + {v_{m+1}^{(o)}}^2) |f'(z)|^2 dA$$

$$- \left( \int_U v_m^{(o)} |f'(z)|^2 dA \right)^2 - \left( \int_U v_{m+1}^{(o)} |f'(z)|^2 dA \right)^2 \geq 2A.$$

*Equality occurs in both inequalities if and only if $f(z) = z$.*

**Theorem 3** *Let $D$ be a simply connected domain in the plane with area $A < \infty$ and maximal conformal radius 1. Then, for any $n \geq 2$ we have*

$$\sum_1^n \frac{1}{\mu_j} \geq \sum_1^n \frac{1}{\mu_j^{(o)}},$$

*where $\mu_j^{(o)}$ are the free membrane eigenvalues of the unit disk. Equality occurs if and only if $D$ is the unit disk.*

### 2.3 Sums of all reciprocal eigenvalues

#### 2.3.1 Fixed membrane

**Theorem 4** *Let $f$ be a conformal mapping from the unit disk $U_1$ onto the domain $D$ with the area $A$, then it holds*

$$\int_{U_1}\int_{U_1} G^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2 dA_z dA_\zeta = \sum_{j=1}^{\infty}\frac{1}{\lambda_j^2},$$

*where $G(z,\zeta)$ denotes Green's function of the unit disk.*

**Theorem 5**

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^2} = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(A_{m,l}+B_{m,l})a_{0,m}a_{0,l} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}(C_{k,m,l}+D_{k,m,l}+\sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}E_{k,m,l},$$

*where the coefficients $A, B, C, D, E$ are known and*

$$|f'(r,\varphi)|^2 = \sum_{n=0}^{\infty}a_{0,n}r^n \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{m,n}\cos m\varphi + b_{m,n}\sin m\varphi)r^m$$

**Examples**

**1. Disk** $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \frac{4}{2 \cdot 4^3} + \sum_{n=1}^{\infty} \frac{8}{4(2n+4)(4n+4)(2n+2)} = \frac{\pi^2}{48} - \frac{5}{32}$

**Remark 1** *Let $\lambda_j(n)$ be zeros of the Bessel function $J_n$ with the same order $n$ (Rayleigh, Scientific papers, 1899).*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j(n)^2} = \frac{1}{16(n+1)^2(n+2)}.$$

**2. Kardioide**

$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \frac{3}{64}\pi^2 - \frac{551}{1536}$

Similar results are obtained for the image of the unit disk by $f_n(z) = z + \frac{1}{n} z^n, n = 2, 3, ...$
.

## 3. Regular n-gone

The conformal mapping $f_n$ of the unit disc onto a regular n-gone is well-known

$$f_n(z) = \int^z \frac{d\zeta}{(1-\zeta^n)^{2/n}}, n \geq 3,$$

The following table contains some numerical results

| $n$ | $\sum_{j=1}^{\infty} \lambda_j^{-2}$ |
|---|---|
| 4 | 0.0514 |
| 5 | 0.0498 |
| 6 | 0.0496 |
| 7 | 0.0494 |
| 8 | 0.0494 |

An **open problem** is:

Prove that among all n-gones with the same maximal conformal radius the regular n-gone has the least value for the sum above.

### 2.3.2 Free membrane

**Theorem 6** *If $D$ is a simply connected sufficiently smooth bounded domain with the area $A = \int_U |f'(z)|^2 dA_z < \infty$. Then for the eigenvalues of the free membrane it holds*

$$\int_U \int_U \Big(N_f(z,\zeta) - AC\Big)^2 |f'(z)|^2 |f'(\zeta)|^2 dA_z dA_\zeta = A^2 \sum_{j=2}^{\infty} \frac{1}{\mu_j^2},$$

*where* $C = -\frac{1}{A^2} \int_U \int_U N(z,\zeta) |f'(z)|^2 |f'(\zeta)|^2 dA_z dA_\zeta$.