Sharp isoperimetric inequalities in the plane*

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Queen Dido Conference
Carthage, May 24 - 29, 2010

*Joint works with A. Alvino and C. Nitsch

Isoperimetric inequality

Let $E \subset \mathbb{R}^n$ be a measuble set of finite measure, then:

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In other words, it holds:

$$Per(E^\sharp) \leq Per(E),$$

where $E^\sharp$ is a ball such that $|E^\sharp| = |E|$.

[E. De Giorgi, 1954]

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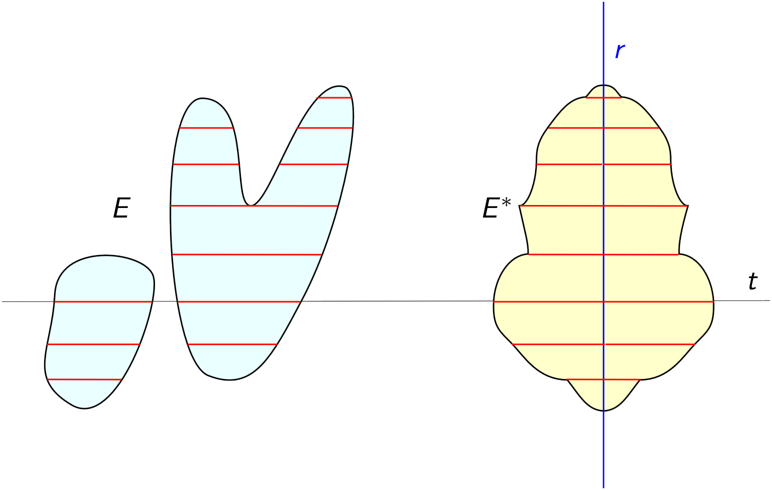
When $n = 2$ it reads as:

$$2\sqrt{\pi|E|} \le Per(E).$$

Steiner symmetrization



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- area is left invariant

$$|E| = |E^*|;$$

- the perimeter decreases

$$Per(E) \geq Per(E^*).$$

A classical question

Let $E \subset \mathbb{R}^2$ be a bounded set of finite perimeter with measure $|E|$ and let us suppose that

$$Per(E) - 2\sqrt{\pi|E|} \qquad \text{"is small"}.$$

Is it possible to say that E is "close" to a circle?

[F. Bernstein, 1905], [T. Bonnesen, 1921, 1924, 1929], [R. Osserman, 1979], [B. Fuglede, 1989], [H. Groemer - R. Schneider, 1991], [S. Campi, 1992], [R.R. Hall, 1992], [R.R. Hall - W.K. Hayman, 1993], [N. Fusco - F. Maggi - A. Pratelli, 2008]

Bonnesen type inequalities

In 1924 Bonnesen has proved the following inequality concerning convex sets $E \subset \mathbb{R}^2$:

$$Per(E)^2 - 4\pi|E| \geq 4\pi d^2,$$

where d is the width of the smallest circular annulus which contains ∂E.

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$$Per(E)^2 - 4\pi|E| \geq \lambda(E)$$

- $\lambda(E) \geq 0$;
- $\lambda(E) = 0$ only when E is a disk;
- $\lambda(E)$ measures the asymmetry of E.

[R. Osserman, 1979]

Some definitions

Fraenkel asymmetry of a set

$$\alpha(E) = \min_{x \in \mathbb{R}^n} \frac{|E \Delta B_r(x)|}{|E|}, \qquad |B_r(x)| = |E|,$$

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Hausdorff asymmetry of a set

$$\delta(E) = \min_{x\in\mathbb{R}^n} d_H(E, B_r(x)), \qquad |B_r(x)| = |E|,$$

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Isoperimetric deficit of a set

$$\Delta P(E) = \frac{Per(E) - Per(B_r)}{Per(B_r)}, \qquad |B_r| = |E|.$$

Question

The isoperimetric inequality can be written as

$$\Delta P(E) \geq 0$$

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Is it possible to estimate $\alpha(E)$ or $\delta(E)$ in terms of $\Delta P(E)$?

For instance, is it possible to obtain an inequality in the form

$$\Delta P(E) \geq f(\alpha(E))$$

for some function $f \geq 0$?

Quantitative isoperimetric inequalities

Theorem *For every convex set* $E \subset \mathbb{R}^n$ *with* $0 < |E| < \infty$ *it holds:*

$$\delta(E) \leq \begin{cases} c(\Delta P(E))^{1/2} & \textit{for } n = 2 \\ c\left(\Delta P(E) \log \frac{1}{\Delta P(E)}\right)^{1/2} & \textit{for } n = 3 \\ c(\Delta P(E))^{\frac{2}{n+1}} & \textit{for } n \geq 4 \end{cases}$$

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Theorem *There exists a constant $c(n)$ such that for every Borel set $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$ it holds:*

$$\alpha(E) \le c(n) \Delta P(E)^{1/2},$$

or, recalling that $Per(B_r) = n\omega_n^{1/n}|E|^{1-1/n}$,

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A quantitative isoperimetric inequality in the plane

Theorem *If $E \subset \mathbb{R}^2$ is convex it holds:*

$$\Delta P(E) \geq \frac{\pi}{8(4-\pi)}\alpha(E)^2 - c\alpha(E)^3,$$

where c is an absolute constant, while the constant $\frac{\pi}{8(4-\pi)}$ *is sharp.*

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Remark The proof is based on an inequality concerning the coefficients of the Fourier series of the support function of the set E. The constant c is not known and it could also be zero.

Applications of quantitative isoperimetric inequalities

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- Stability results for comparison inequalities for solutions of PDE's
- Stability results for radial symmetry of solutions to overdetermined boundary value problems

Aftalion, Bianchi, Brandolini, Busca, Cianchi, Egnell, Esposito, A. Ferone, Fusco, Henrot, Maggi, Nitsch, Pratelli, Reichel, Rosset, Salani, C. Trombetti, ...

An isoperimetric inequality for the deficit

Theorem *Let* $E \subset \mathbb{R}^2$ *be convex. There exists an unique set* $\widehat{E}$ *in a certain class* $\mathfrak{C}$ *such that*

$$\alpha(E) = \alpha(\widehat{E}) \quad \textit{and} \quad \Delta P(E) \geq \Delta P(\widehat{E}) \quad (\textit{i.e., } Per(E) \geq Per(\widehat{E})).$$

[A. Alvino - V.F. - C. Nitsch, JEMS (to appear)]

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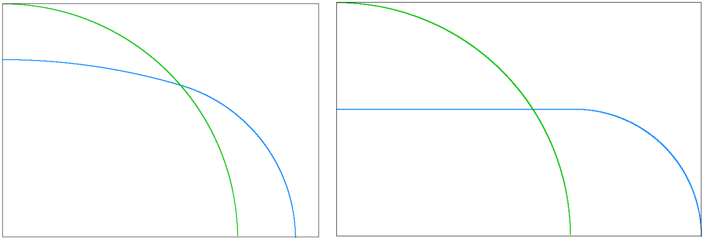
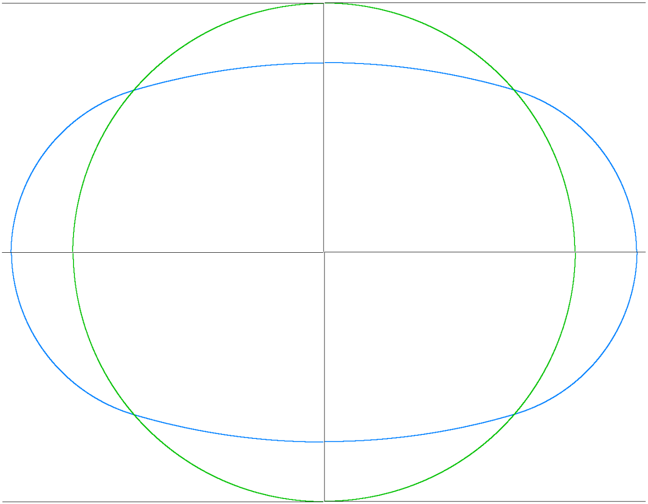


A family in the set $\mathfrak{C}$



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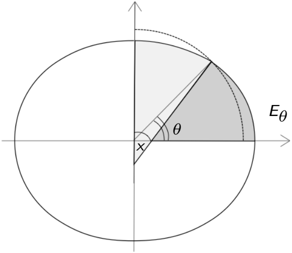
Analytic description of the family $\mathfrak{C}$



Let us consider a set in $\mathfrak{C}$ as in the figure, it is possible to write ΔP and α as follows:

$$\Delta P(E_\theta) = \frac{2}{\pi}\left(\left(\frac{\sin\theta}{\cos x}\left(\frac{\pi}{2} - x\right) + \frac{\cos\theta}{\sin x}x\right) - \frac{\pi}{2}\right)$$

$$\alpha(E_\theta) = \frac{4}{\pi}\left(\frac{\sin^2\theta}{\cos^2 x}\left(\frac{\pi}{2} - x - \sin x\cos x\right) - \theta + \sin\theta\,\cos\theta\right)$$

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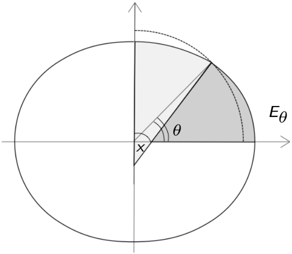

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where θ and x satisfy the equality:

$$\frac{\sin^2\theta}{\cos^2 x}\left(\frac{\pi}{2}-x\right)+\frac{\cos^2\theta}{\sin^2 x}x-\frac{(\cos\theta-\tan x\,\sin\theta)^2}{\tan x}=\frac{\pi}{2}$$

which can written as:

$$\tan\theta = \frac{1}{\tan x}\,\frac{\pi\sin^2 x-2x+2\sin x\cos x}{2x-\pi\sin^2 x+2\sin x\cos x}$$

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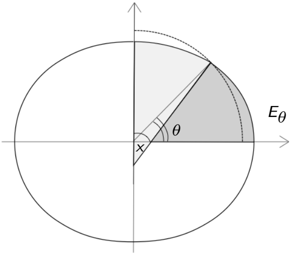

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It is possible to obtain the following expansion in $\theta = \pi/4$:

$$\frac{\Delta P(E_\theta)}{\alpha(E_\theta)^2} = \frac{\pi}{8(4-\pi)} - \left(\frac{\pi}{4}\right)^3 \frac{(16-5\pi)(14-3\pi)}{6(4-\pi)^4(\pi-2)}\alpha(E_\theta)^2 + O\left(\alpha(E_\theta)^4\right)$$

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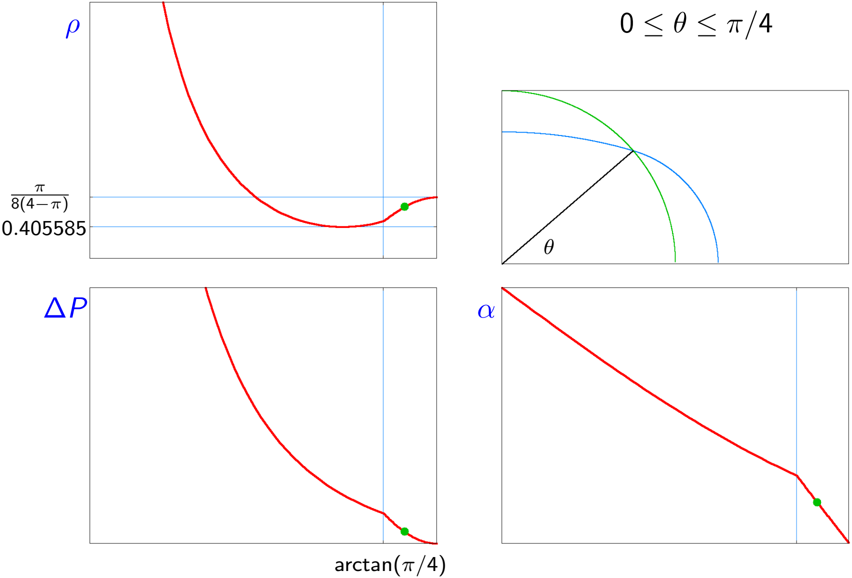

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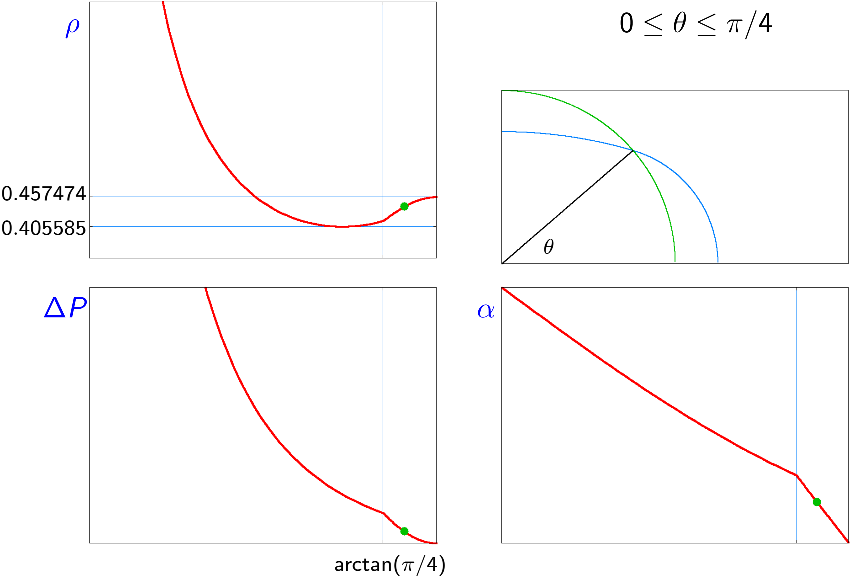

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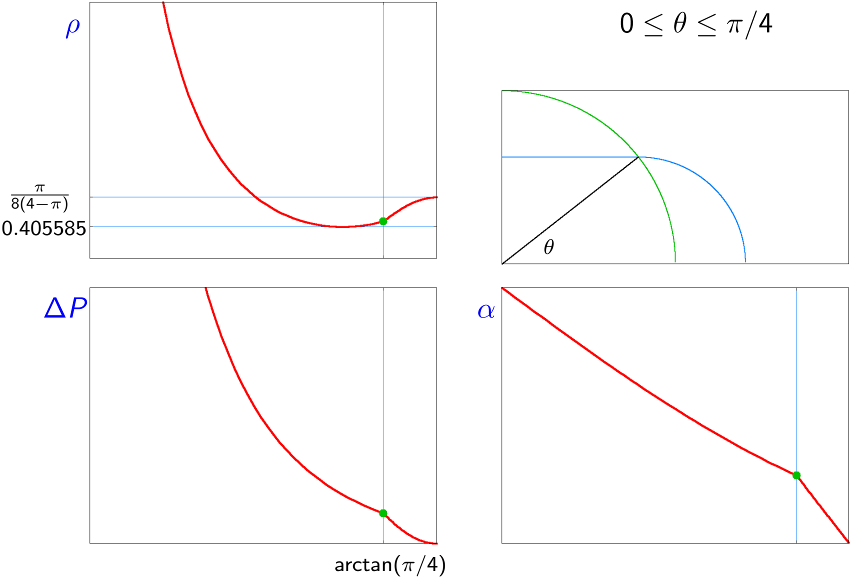

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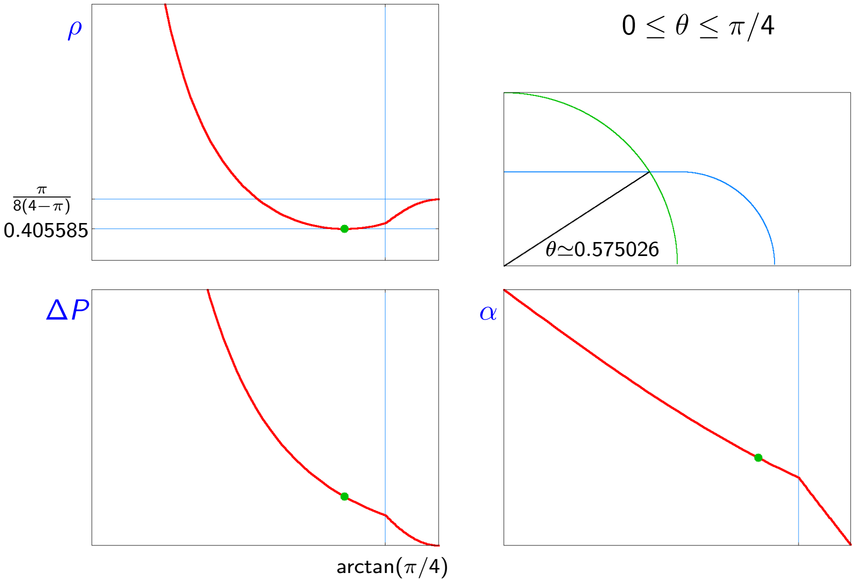

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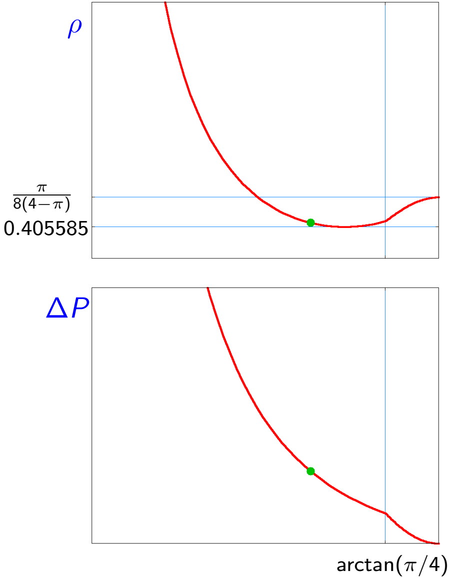

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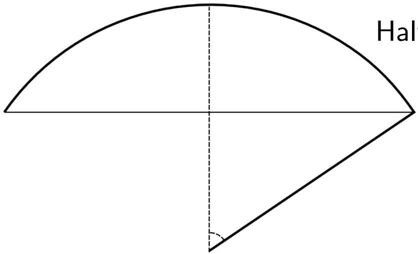
The family $\mathfrak{L}$

We say that a convex set E belongs to the family $\mathfrak{L}$ (*lenses*) if E is symmetric with respect to a straight line and the part of it which stays on one side of the line coincides with a circular segment.



Half of a set in $\mathfrak{L}$

Analytic description of the family $\mathfrak{L}$

It is possible to parametrize the sets in $\mathfrak{L}$ with a given fixed measure in terms of the Hausdorff asymmetry index. So, we can denote by Y_δ the set in $\mathfrak{L}$ such that:

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$$4\pi^2 = \lim_{\delta\to 0} \frac{P(Y_\delta)^2 - 4\pi|Y_\delta|}{\delta^2} > \frac{P(Y_\delta)^2 - 4\pi|Y_\delta|}{\delta^2} > \lim_{\delta\to +\infty} \frac{P(Y_\delta)^2 - 4\pi|Y_\delta|}{\delta^2} = 16.$$

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Corollary *Every convex set $\Omega \in \mathbb{R}^2$ satisfies the inequalities*

$$P(\Omega)^2 - 4\pi|\Omega| \geq 16\delta(\Omega)^2,$$

$$P(\Omega)^2 - 4\pi|\Omega| \geq \delta(\Omega)^2(4\pi^2 - H(\delta(\Omega))),$$

for some positive $H(\delta) = O(\delta)$.

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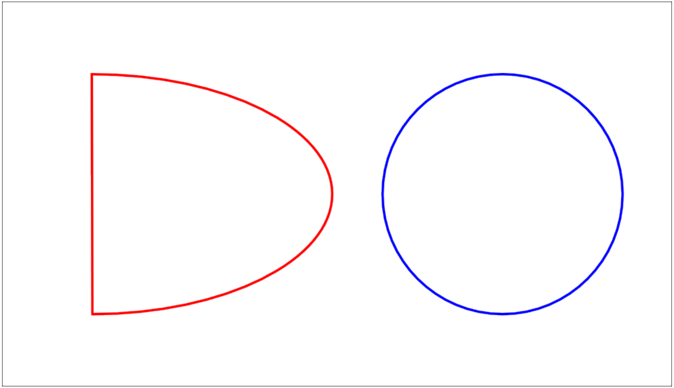
- the asymmetry index does not change;
- the isoperimetric deficit does not increase;
- the symmetry properties "improve".

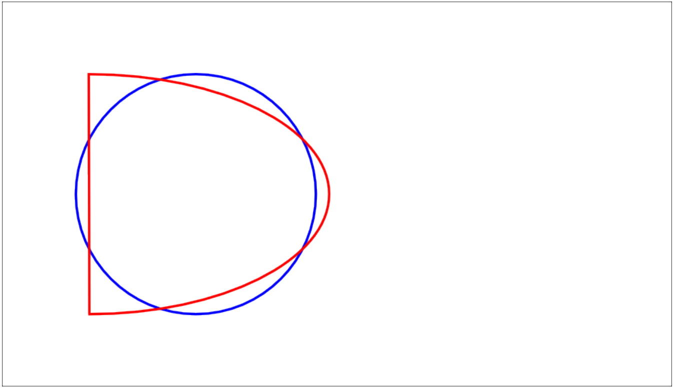
Steiner symmetrization cannot be used as the first step

Let us consider a set and the circle having the same measure.



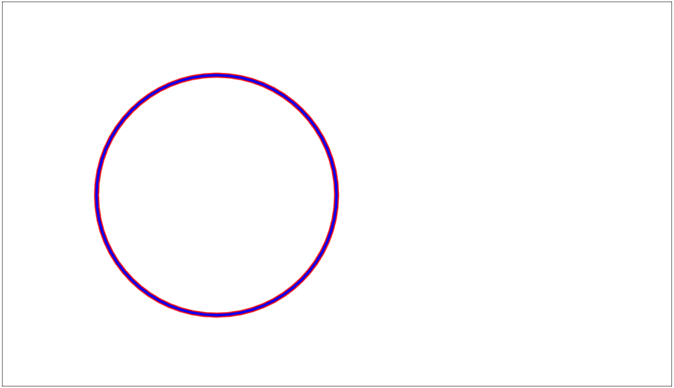
Steiner symmetrization cannot be used as the first step

Let us consider the circle in the optimal position.



Steiner symmetrization cannot be used as the first step

Let us symmetrize the set with respect to the vertical axis.



The reduction to a 2-symmetric set

Let $E \subset \mathbb{R}^2$ be convex. We build a 2-symmetric set $\widetilde{E}$ such that:

$$\alpha(E) = \alpha(\widetilde{E}), \qquad \Delta P(E) \geq \Delta P(\widetilde{E}). \tag{1}$$

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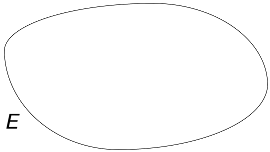

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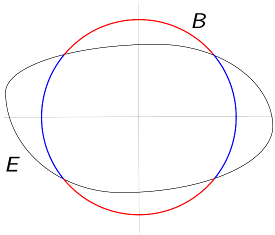

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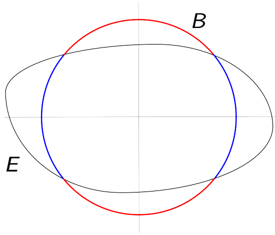
The simplest case is shown in the figure: the boundary of E is cut by the boundary of B exactly in four pieces, two pieces are external to B, two pieces are internal.



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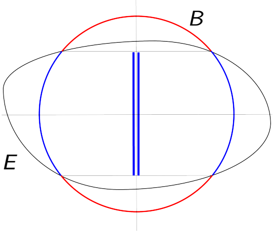
Using the fact that B realizes the minimum in Fraenkel asymmetry, one immediately has that the red and the blue arcs in the figure have to be equal.



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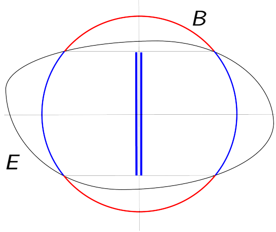

At this point we can apply Steiner Symmetrization with respect to the coordinate axes obtaining a set $\widetilde{E}$ which satisfies (1).

The reduction to a 2-symmetric set

Remark We observe that the conclusion about the equality of the circular arcs comes from the property about the projections of such arcs described below.

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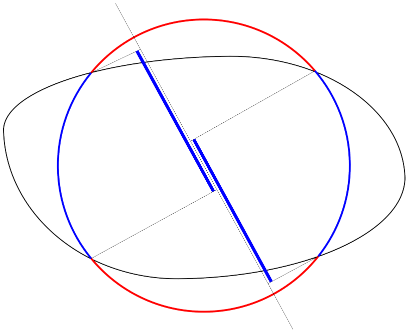

The reduction to a 2-symmetric set (continued)

Let us analyze the general case. Typical situations are given in the figures.

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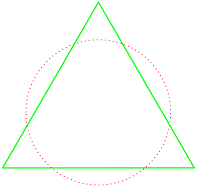

E is an equilateral triangle

The reduction to a 2-symmetric set (continued)

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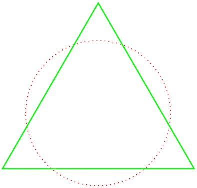

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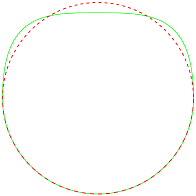

∂E has a coincidence part

The reduction to a 2-symmetric set (continued)

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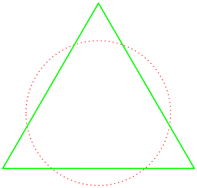

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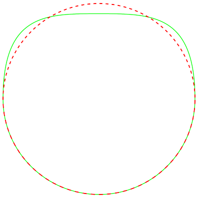

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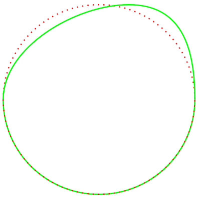
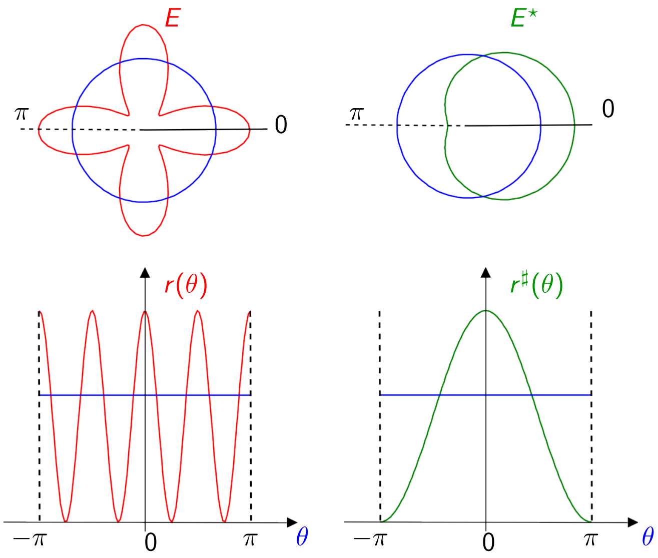


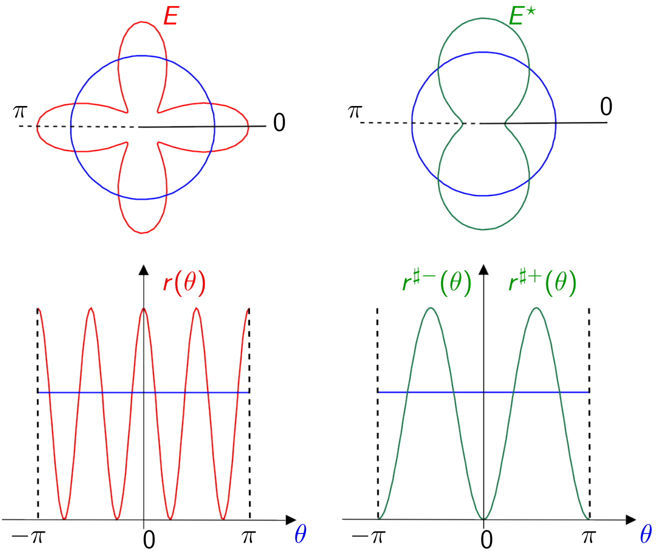
∂E has one coincidence part, one external part and one internal part

How is it possible to reduce these general situations to the simple one we have already studied?

Circular symmetrization



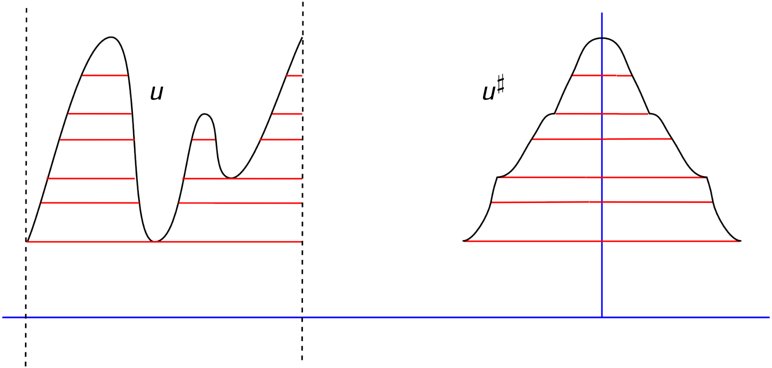
Circular symmetrization



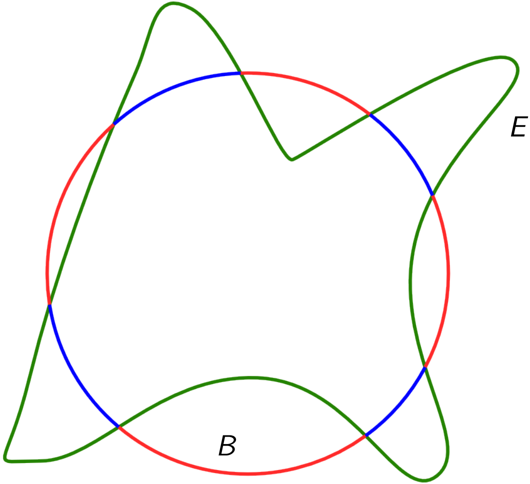
Schwarz symmetrization for functions of one variable

$$||u||_p = ||u^\sharp||_p, \qquad ||u'||_\infty \geq ||(u^\sharp)'||_\infty$$

if $\#\{u = t\} \geq 2$



A lemma about projections



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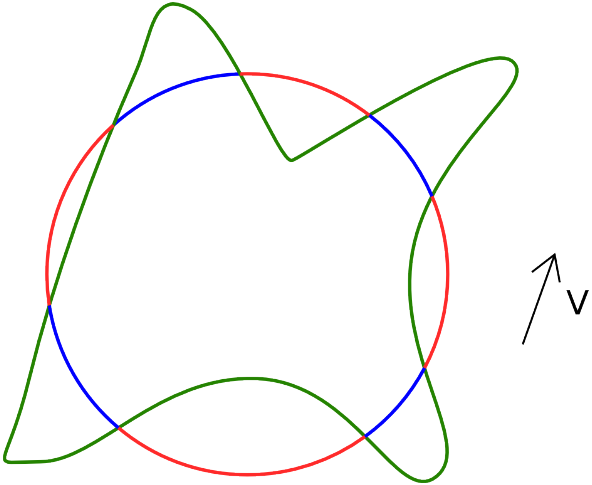

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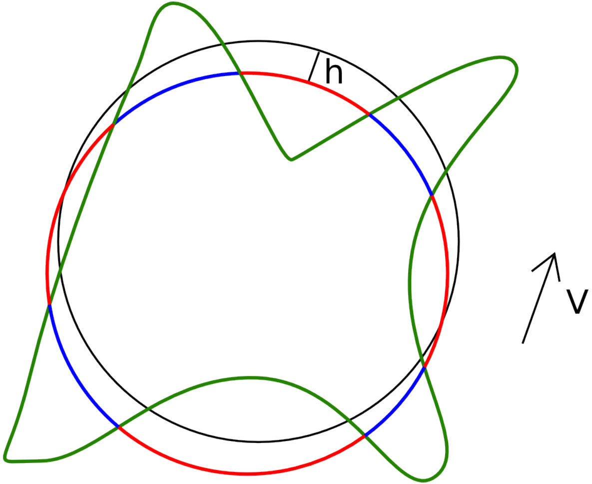

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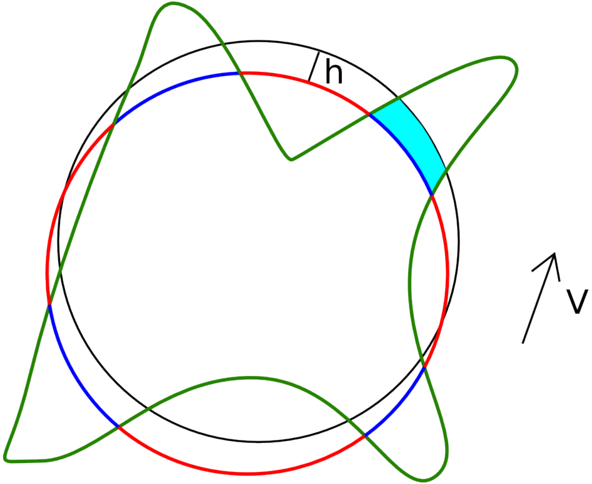

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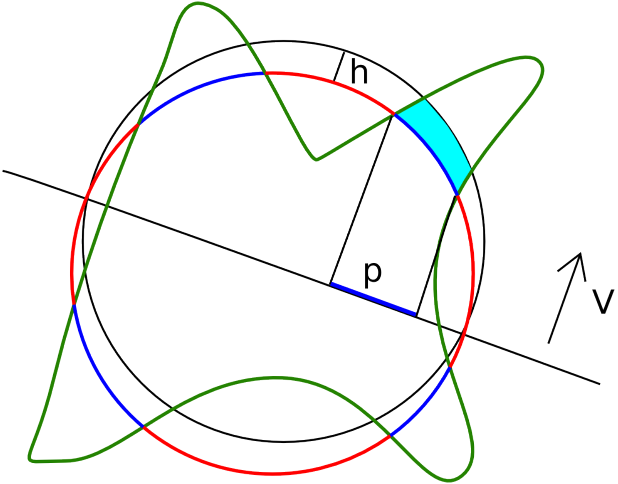

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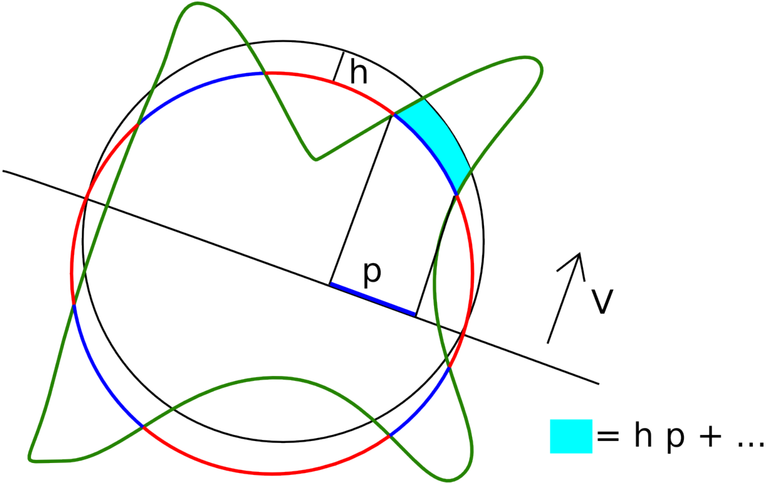

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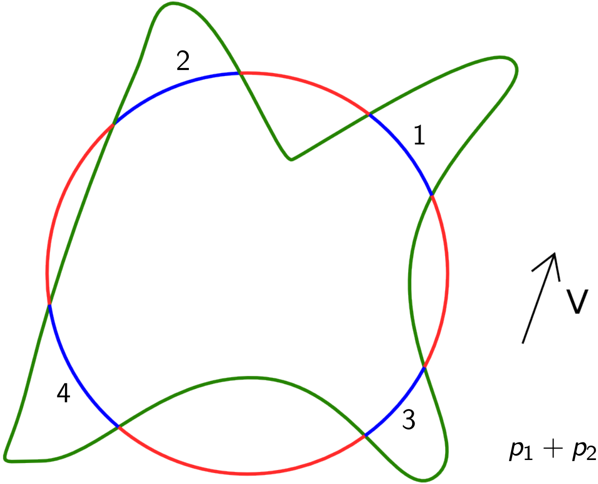

$p_1 + p_2 \leq p_3 + p_4$

The reduction to a 2-symmetric set (continued)

General case

Let us introduce the following notation:

- $\Gamma_+ = \partial B \cap E$;
- $\Gamma_- = \partial B \setminus \bar{E}$;
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- L length of the arcs.

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Lemma *If $L(\Gamma_0) = 0$, for every $i \in \mathfrak{I}$ and $k \in \mathfrak{K}$ it holds:*

$$L(\gamma_+^i) \leq \sum_{j \neq i} L(\gamma_+^j),$$

$$L(\gamma_-^k) \leq \sum_{j \neq k} L(\gamma_-^j).$$

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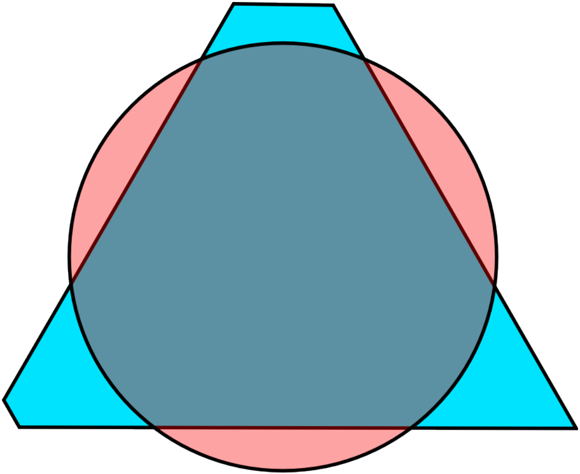
Lemma *There exists $0 \le \bar{\eta} \le 1$ such that, for every $i \in \mathfrak{I}$ e $k \in \mathfrak{K}$, it holds:*

$$L(\gamma_+^i) \le \sum_{j \neq i} L(\gamma_+^j) + \bar{\eta} L(\Gamma_0)$$

$$L(\gamma_-^k) \le \sum_{j \neq k} L(\gamma_-^j) + (1 - \bar{\eta}) L(\Gamma_0).$$

The reduction to a 2-symmetric set (continued)



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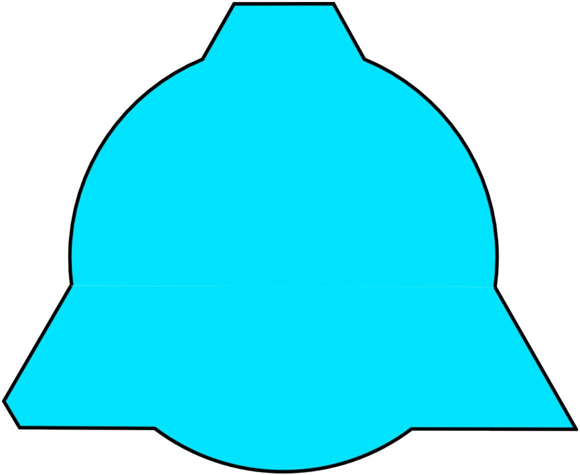

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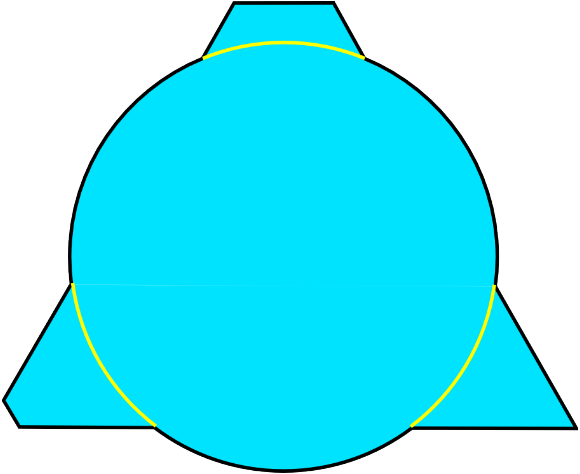

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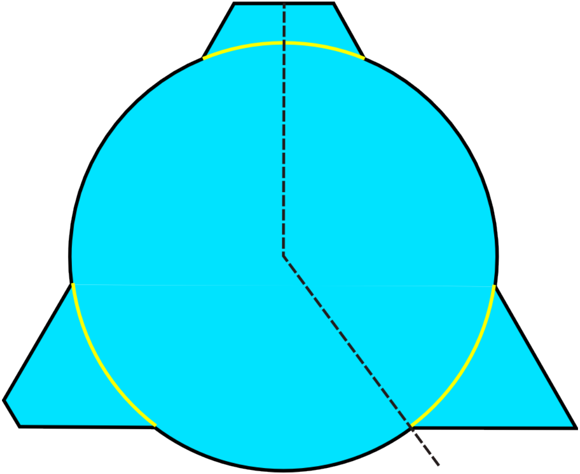

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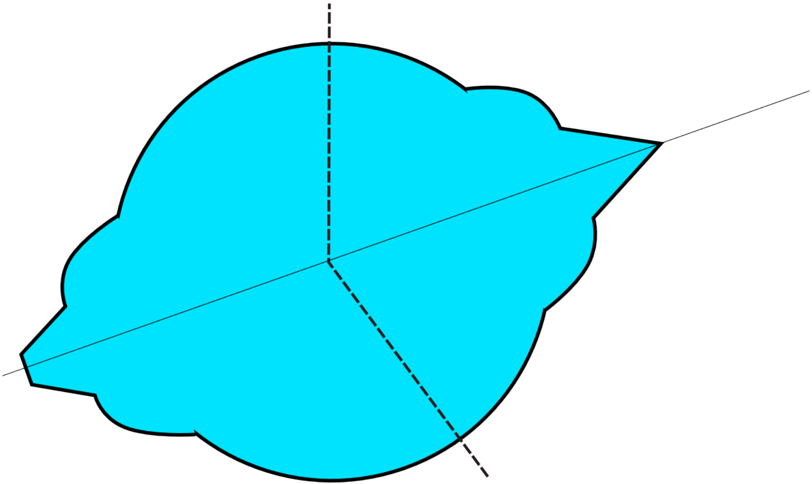

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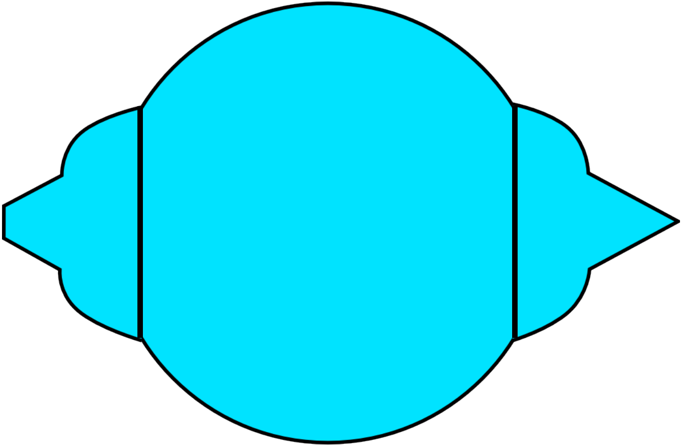

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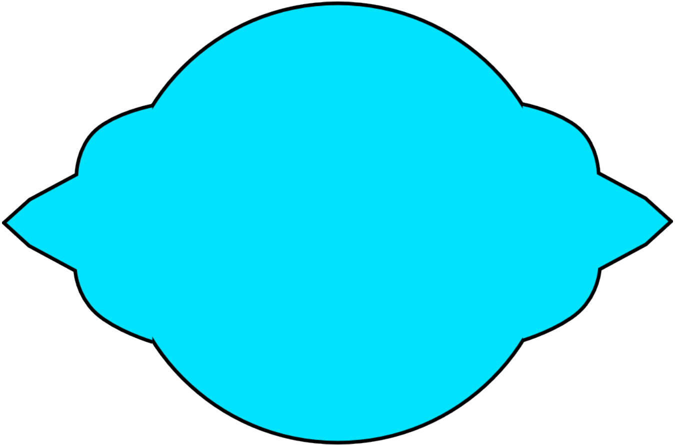

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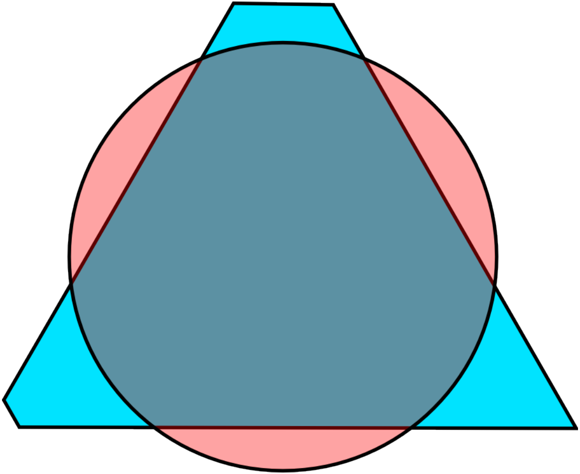

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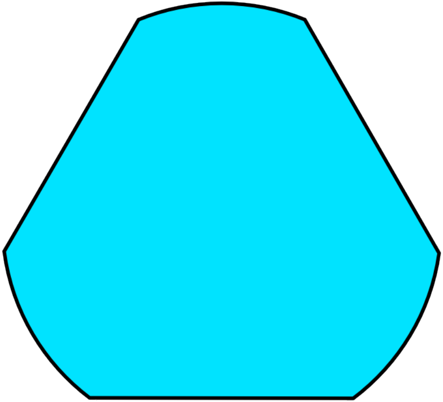

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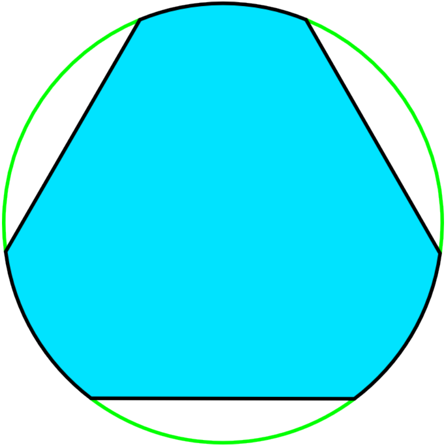

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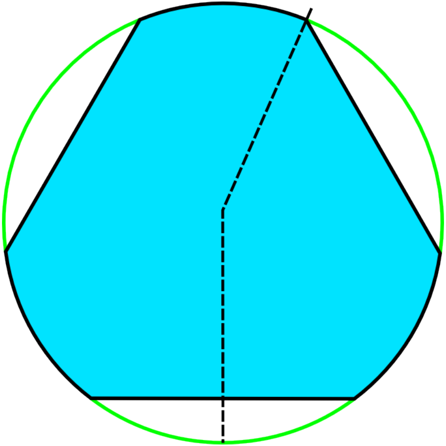

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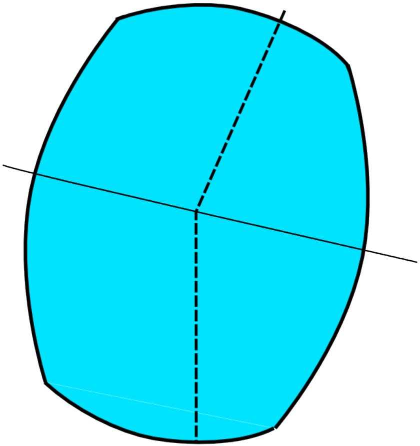

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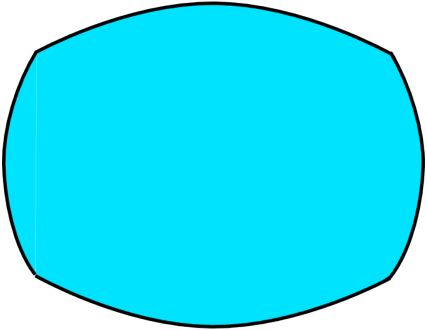
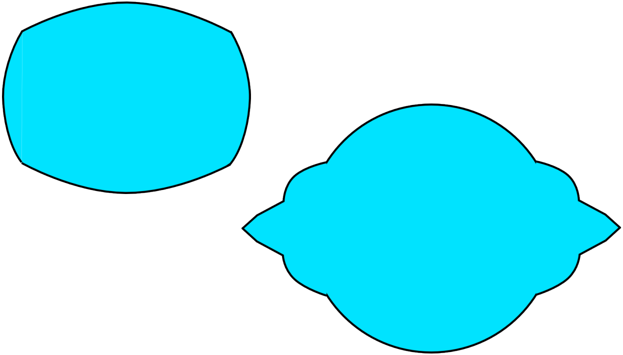


The reduction to a 2-symmetric set (continued)



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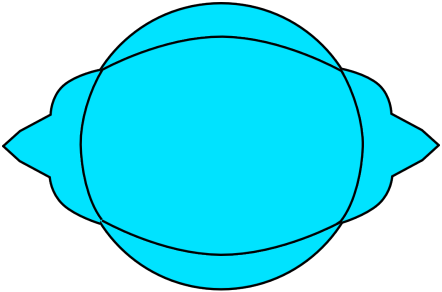
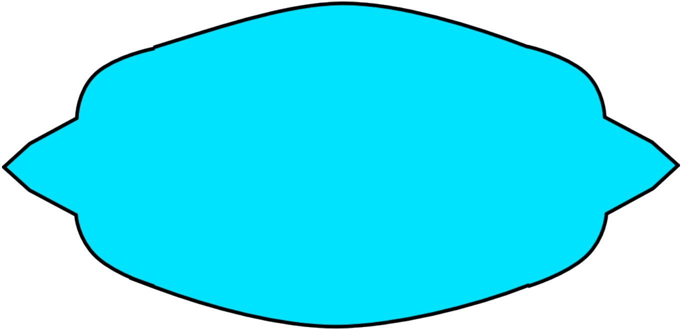
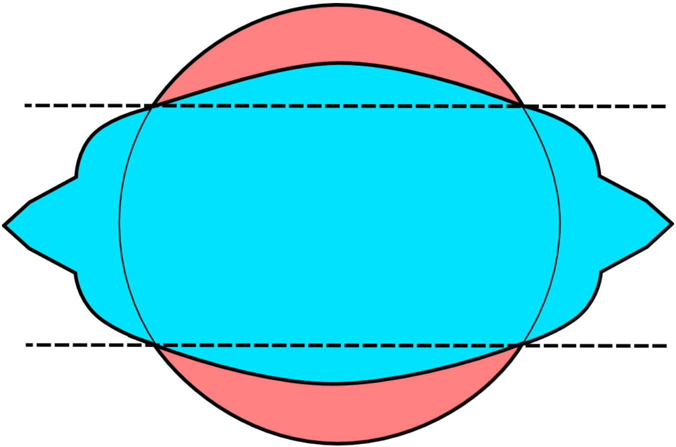


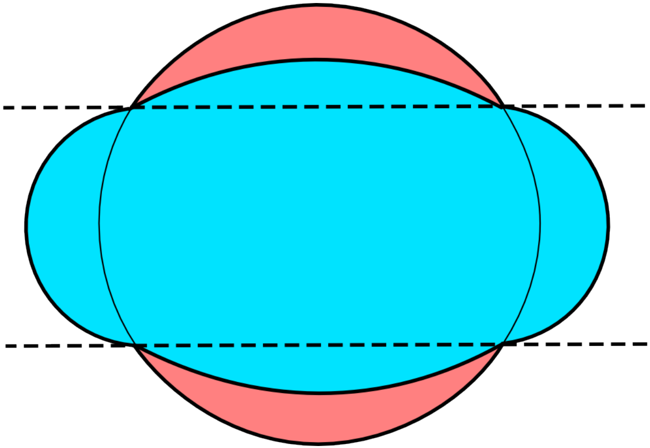
The reduction to a 2-symmetric set (continued)



The family $\mathfrak{B}$



End of the proof



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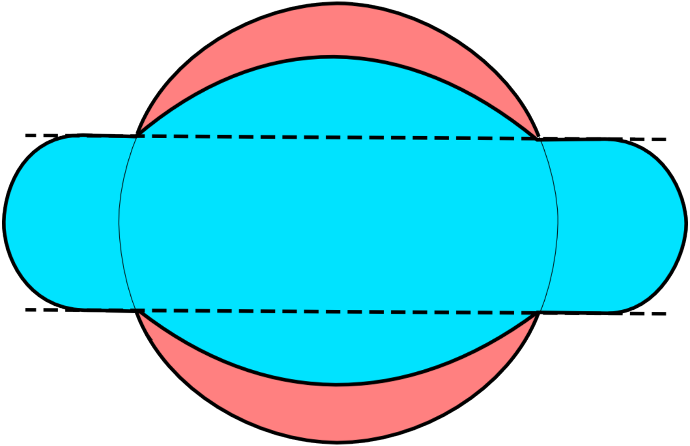
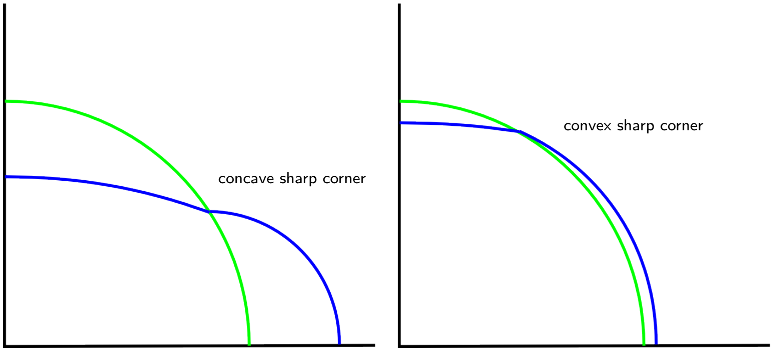


Regularity C^1

If the boundary is not of class C^1, there are two possible cases.



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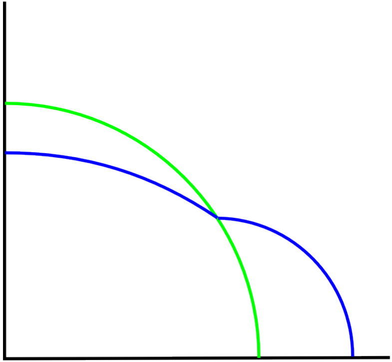

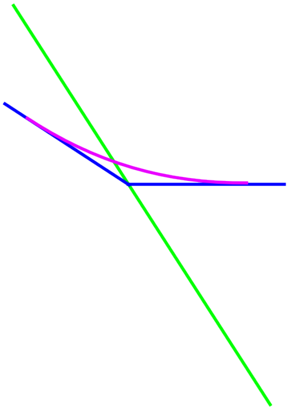

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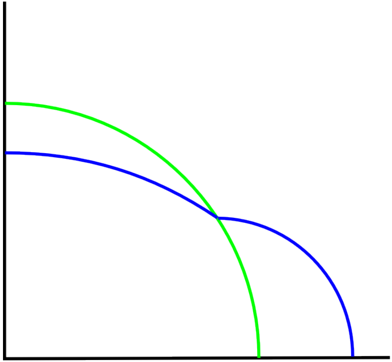

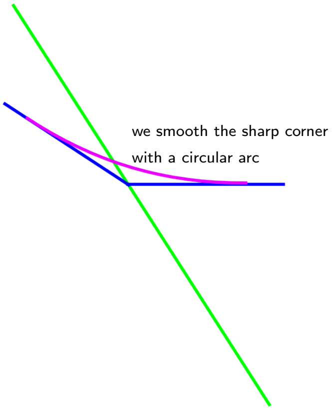

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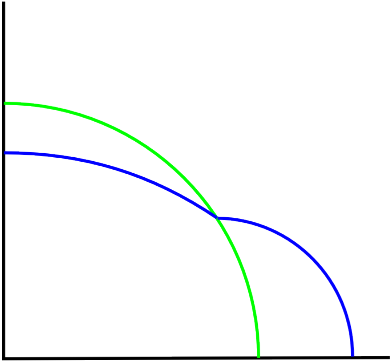
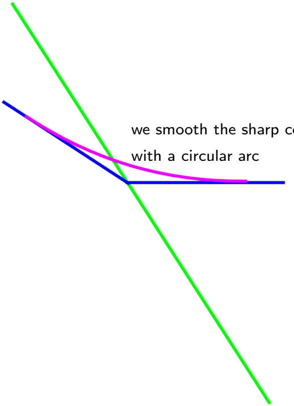
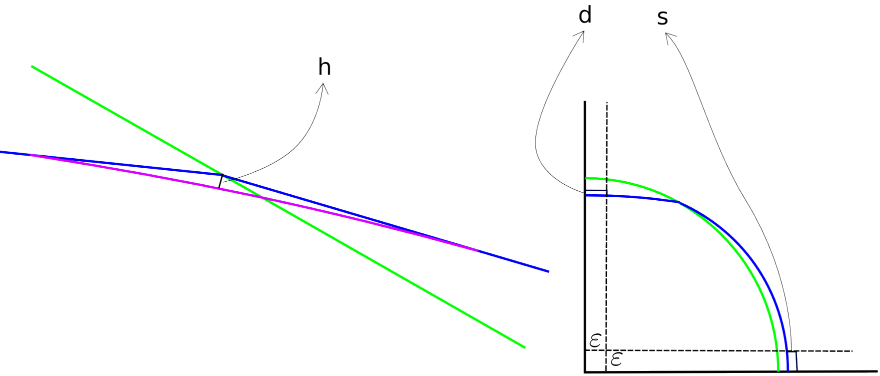




It is not difficult to restore the initial values of area and asymmetry index without increasing the perimeter.

Regularity C^1



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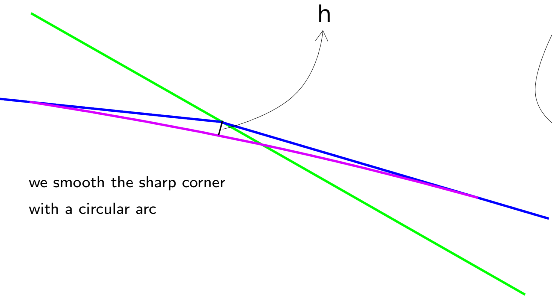
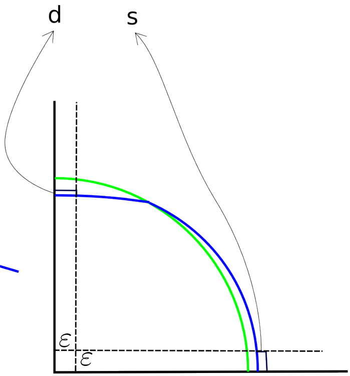


we smooth the sharp corner
with a circular arc



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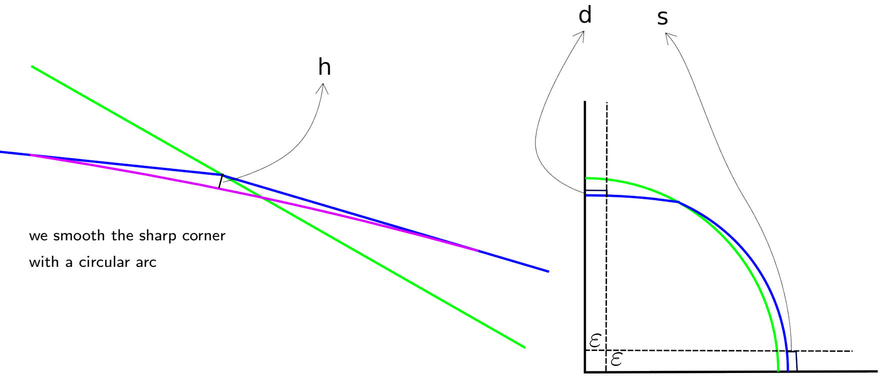

As in the previous case, it is possible to choose ε, d and s in such a way that the initial values of area and asymmetry index are restored without increasing the perimeter.