Around Eigenvalues of some elliptic problems

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I. Mathematics in Tunis in 1968-70

"Tunis naguère (et aujourd'hui)" Zoubeir Turki-

II. The counting function

III. The partition function

IV. **Domains with fractal boundary**

I. MATHEMATICS IN TUNIS in 1969-1970

(A local and more recent history!)

Location:

In 1968, the university was located in the downtown **Place de la Monnaie**, (near Ave de Paris). The Dean was Adnan Zmerli (physicist).

Then in 1969 we first moved to what is now **ENIT**, (Ecole Nationale d'Ingenieurs de Tunis)

During spring 1970 the faculty of sciences was finished.

The hill (which is now El Menzah X or ....) was almost empty; there were three buildings (these two ones and the RTT). There was only one tree by the TV and there was only one road to reach the "campus". Around 1969, Cite Carnoy (downhill) was built.

Raoued - Gammarth After the "Lagon Bleu" it was completely empty. Except 2 restaurants and one center for children there was nothing on this huge beach untill Raoued village. Around 1989 they started the construction of some hotels... And now...

People MS Baouendi and his collegues...: Mainly Fatma Moalla, Khelifa Harzallah, Mohamed Amara, attracted several mathematicians. We saw the coming of

JM Bony, J.Faraut, C.Goulaouic, P.Grisvard, B.Malgrange, JP Ramis, J.Chazarin, G.Ruget, J.Camus, P.Bolley, ...

Also for shorter visits:
L.Hörmander, JP Kahane, P.Blanchard, ...

The young tunisians: Moncef Hamza,... Houcine Chebli (ex Ouergemi) Khelifa Trimeche,...

Tunis was an excellent department in analysis, (PDE and potential theory mainly). All the members of this math department were young and enthousiastic.

"Can one hear the shape of a drum?" (by M.Kac) was the first paper that I studied under the supervision of Jacques Faraut in Tunis.

Now there are more than 100 mathematicians attending the annual conference of SMT (Société Mathématique de Tunisie) and also many universities, ISET, ...

II. RECALLS ON THE COUNTING FUNCTION

Let Ω be a bounded domain in $I\!R^n$ with boundary $\partial\Omega$ and consider the EVP

$$(EV) \qquad -\Delta u = \lambda u \text{ in } \Omega; \;\; u|_{\partial\Omega} = 0.$$

There exists a countable set of eigenvalues:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_k \leq ...$$

each eigenvalue being repeated according to (algebraic) mutiplicity.

For a given positive λ, the *"counting function"* is

$$N(\lambda, -\Delta, \Omega) = N(\lambda) := \#\{(0 <)\lambda_k < \lambda\}.$$

II. 1. The rectangle and Gauss estimate

Consider now a rectangle: $\mathcal{R} = (O, a) \times (O, b) \subset \mathbf{R}^2$; For that domain the solutions to (EV) are :

(1) $u(x, y) = sin(p\pi x/a) sin(q\pi y/b),\ p \in \mathbf{N}^*;\ q \in \mathbf{N}^*.$

The associated eigenvalues are $\lambda_{p,q} = (p.\pi/a)^2 + (q.\pi/b)^2$.

We make use of Gauss estimates (1801) on the number of pair of integers inside the quater of ellipse

$$\mathcal{N}(\lambda/\pi^2) := \#\{(p, q) \in I\!N^* \times I\!N^* / (\frac{p\pi}{a})^2 + (\frac{q\pi}{b})^2 < \lambda\};$$

this number is proportional to the quarter of the area of that ellipse, and hence for the counting function :

(2) $N(\lambda, -\Delta, \mathcal{R}) \simeq (4\pi)^{-1}.a.b.\lambda,$ as $\lambda \to \infty.$

II. 2. A bounded domain and Weyl's estimate

Now $\Omega \subset \mathbf{R}^n$ is a sufficiently "smooth" bounded domain with volume $|\Omega|$; *H.Weyl* (1911) has shown that [We]:

(3) $N(\lambda, -\Delta, \Omega) \simeq (2\pi)^{-n}\omega_n|\Omega|\lambda^{n/2}$, as $\lambda \to +\infty$,

with ω_n the volum of the unit ball in $I\!R^n$.

Notation: From now on set

$$(W) \qquad W(\lambda, \Omega) := (2\pi)^{-n}\omega_n|\Omega|\lambda^{n/2}$$

the **"Weyl term"**.

II. 3. Some extensions

Estimate (3) holds also

for irregular domains ($e.g.$ with cusps) (JF + G.Métivier, 1973),

for unbounded domains with finite measure.

There are also results for Schrödinger operators, for problems defined on unbounded domains

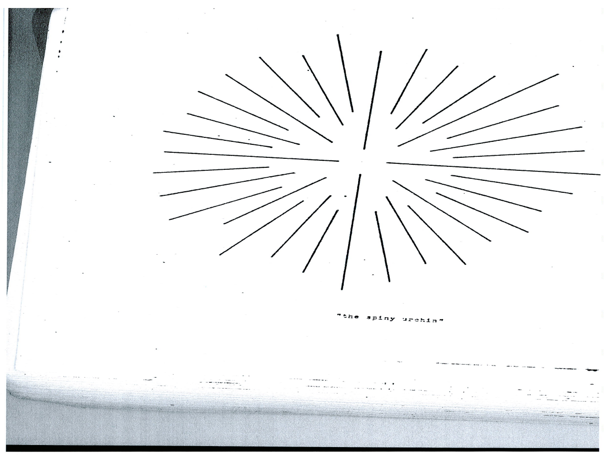
For weighted elliptic problems of Schrödinger type:

$$\mathcal{L}u + q(x)u = \lambda p(x)u$$

defined on $I\!R^n$, we derive two kinds of behavior (as Weyl or as Schrödinger) if $p/q \to 0$ at infinity.

For the *spiny urchin* (C.Clark), as $\lambda \to \infty$:

$$N(\lambda, -\Delta) \simeq \lambda Ln^2(\lambda)$$



After 1911, for the "classical" problem above, a challenge was to obtain a more precise estimate in particular the second term of the counting function.

II.4. The Remainder term

We study now, as $\lambda \to +\infty$, the *"remainder term"* :

$$N(\lambda, -\Delta, \Omega) - W(\lambda, \Omega).$$

Under some conditions on the **symmetries** of the domain, and on the **smoothness** of the boundary, then

(4)

$$N(\lambda, -\Delta, \Omega) = W(\lambda, \Omega) - \gamma_n L(\partial\Omega)\lambda^{(\frac{n-1}{2})} + o(\lambda)^{(\frac{n-1}{2})}$$

as $\lambda \to +\infty$.

Here γ_n is a constant depending only on n;

$L(\partial\Omega)$ is the $n-1$ measure (length) of $\partial\Omega$.

For these estimates 2 methods:

Fourier Integral operators (see $e.g.$ Ivrii, Sjöstrand, Helffer...)

Dirichlet Neumann bracketing (JF, Métivier, Edmunds and Evans,...)

Dirichlet Neumann bracketing and some inequalities

■ $N(\lambda, -\Delta, \Omega)$ the counting function for the Dirichlet Laplacian increases with the domain Ω.

■ $N(\lambda, -\Delta, \Omega) \leq N_N(\lambda, -\Delta, \Omega)$ where $N_N(\lambda, -\Delta, \Omega)$ is the counting function for the Neumann problem.

■ Assume

$$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \text{ with } \Omega_1 \cap \Omega = \emptyset$$

then

$$N(\lambda, -\Delta, \Omega_1) + N(\lambda, -\Delta, \Omega_2) \leq N(\lambda, -\Delta, \Omega) \leq$$

$$N_N(\lambda, -\Delta, \Omega_1) + N_N(\lambda, -\Delta, \Omega_2).$$

III. THE PARTITION FUNCTION

III.1. $Z(t)$

Other functions of eigenvalues are also of interest. For example the *partition function* (Laplace transform of the spectral function) $Z(t,\Omega)$:

(5) $$Z(t,\Omega) := \sum_{j=1}^{\infty} e^{-\lambda_j t} = t\int_o^{\infty} e^{-t\lambda} N(\lambda,\Omega)d\lambda$$

It is the trace of the heat kernel; one has:

(6) $$Z(t,\Omega) = (4\pi t)^{-n/2}(|\Omega|_n + a_1 t + \dots.) \quad \text{as } t \to 0;$$

The terms a_1, a_2, ... appearing in (6) are the measure (length) of the boundary and also the number of holes, the curvature,... (*c.f. e.g., Berger; Gauduchon; Mazet).*

Remark The knowledge of $N(\lambda)$ implies the knowledge of $Z(t)$ but the converse is not true!

III.2 "CAN ONE HEAR THE SHAPE OF A DRUM?" [Kac]

This famous question was raised by *M.Kac* in 1965; indeed, is it possible, for a perfect ear, just by listening at the tones and overtones to determine precisely the shape of a vibrating membrane?

Remark From Weyl's estimate we deduce that the knowledge of the frequencies of Ω implies the knowledge of the volum $|\Omega|$. Nevertheless, if the spectra of the Dirichlet Laplacian on a domain gives many informations on the domain it has been proved that the counting function or the partition function do not determine entirely the geometry of the domain. *Urakawa*. *Gordon, Webb and Wolpert* have constructed planar isospectral domains which are not isometrical.

IV. DOMAINS WITH FRACTAL BOUNDARIES

IV.1 Berry's Conjecture

Recall As $\lambda \to +\infty$

(4)

$$N(\lambda, -\Delta, \Omega) = W(\lambda, \Omega) - \gamma_n L(\partial\Omega)\lambda^{(\frac{n-1}{2})} + o(\lambda)^{(\frac{n-1}{2})},$$

In 1979, *M.V.Berry*, studying the scattering of waves by fractals conjectured that (4) was still valid for Ω, with fractal boundary, just by replacing $n-1$ by h the Hausdorff dimension of the boundary [Be]:

(7) $N(\lambda) = W(\lambda, \Omega) - \gamma_n L(\partial\Omega)\lambda^{h/2} + o(\lambda)^{h/2}$, $\lambda \to +\infty$

IV.2. Hausdorff Dimension:

This dimension was popularized by B.Mandelbrot.

For a given number $\epsilon > 0$ we consider covering of the boundary $\partial\Omega$ by balls B_i with radii $r_i \leq \epsilon$.

For any $t > 0$, set

$$M(t) := \lim_{\epsilon \to 0} (\inf \sum r_i^t).$$

The infimum for all coverings defines the Hausdorff dimension:

$$h := \inf\{t > 0 / M(t) < +\infty\}$$

.

IV.3 Bouligand-Minkowski Dimension:

The Bouligand-Minkowski dimension is the Minkowski dimension extended by *Bouligand* [Bo] to non integer numbers.

For a given $\epsilon > 0$ consider the interior boundary strip

(8) $$\Gamma_\epsilon^i := \{x \in \Omega / d(x, \partial\Omega) < \epsilon\}$$

where $d(.,.)$ is the usual distance in $\mathbb{R}^n$.

For all $t > O$, set

$$M^*(t, \partial\Omega) := \limsup_{\epsilon \longrightarrow 0} \epsilon^{-(n-t)} |\Gamma_\epsilon^i|;$$

the interior Bouligand-Minkowski dimension of $\partial\Omega$, d_i is given by:

$$d_i := \inf\{t > O/M^*(t, \partial\Omega) < +\infty\} =$$

(9) $$n - \liminf_{\epsilon \to 0} \frac{Ln|\Gamma^i_\epsilon|]}{(Ln\epsilon)};$$

Analoguously we can define d_e (exterior dimension) and

(10). $$d = max\{d_i, d_e\}$$

It is easy to verify that for a regular domain, the Bouligand-Minkowski dimension of the boundary is $n - 1$.

This last dimension is often called "box-counting dimension", logarithmic dimension, Kolmogorov entropy , ...

It is often used since it can be computed easily by counting the number of squares with side ϵ which are intersected by the boundary, or the curve.

IV.4. Brossard-Carmona conjecture

In 1985, *J.Brossard R.Carmona*, [BC],exhibited a counterexample to Berry's conjecture; precisely they constructed a domain where $h < d_i \leq d$. Therefore, they suggested to replace h by d in (7), with d the Bouligand-Minkowski dimension of the boundary; their conjecture is:

(11)

$$N(\lambda) = W(\lambda, \Omega) - \gamma_n L(\partial\Omega), \lambda^{d/2} + o(\lambda)^{d/2},\ \lambda \to +\infty$$

IV.5. A first estimate for the remainder term

In 1987, JF with *M.Lapidus* have shown that

(12), $N(\lambda, \Omega) = W(\lambda, \Omega) + O(\lambda^{d/2}) \qquad , \lambda \to +\infty$

where d is the Bouligand-Minskowski dimension of the boundary $\partial\Omega$.

This estimate is obtained by using the "Dirichlet-Neumann bracketing" (e.g. *Courant-Hilbert*, *Reed-Simon)*). We consider a partition of Ω into cubes Q_o with side 1; then a partition of the boundary which is left into cubes with side 2^{-1};. . . . At the k-th step a boundary strip is left; denote it by B_k. inside B_k again we can put cubes Q_k with side 2^{-k}; denote by n_k the number of these cubes.

Obviously by ((8)) (definition of Γ_ε),

$$B_k \subseteq \Gamma_{2^{-k}\sqrt{n}} := \{x \in \Omega / d(x, \partial\Omega) < 2^{-k}\sqrt{n}\}.$$

Therefore,

$$n_k \leq C2^{kd}. \tag{13}$$

For λ fixed, $N(\lambda, -\Delta, Q_p) = 0$ for $p \in I\!N$, large enough that is Q_p small enough. Assume that $K \in I\!N$ is such that for any integer $p > K, N(\lambda, Q_p) = 0$.

By use of "Dirichlet- Neumann bracketing", we get:

$$\sum_{k=1}^{k=K} n_k N(\lambda, Q_k) \leq N(\lambda, \Omega) \leq \tag{14}$$

$$\sum_{k=1}^{k=K} n_k N_N(\lambda, Q_k) + N_N(B_k),$$

where$N_N(\lambda, \omega)$ denotes the counting function for the Neumann problem on ω. Combining Gauss formula in $I\!R^n$, estimates (13) and (14) as well as estimates for the boundary strip B_k by *Fleckinger-Métivier* (1973) we derive the result.

Finally Lapidus has shown that the estimates holds with d_i when $d_i \neq d$.

(12), $$N(\lambda, \Omega) = W(\lambda, \Omega) + O(\lambda^{d_i/2}) \qquad , \lambda \to +\infty$$

where d_i is the interior Bouligand-Minskowski dimension of the boundary $\partial\Omega$.

(12) has been extended to more general problems by several people (*M. van den Berg, Chen Hua, Duplantier, W.D.Evans, . . .*).

IV.6 THE SECOND TERM OSCILLATES .

As for smooth domains, it is then natural to try to get a more precise estimate for the second term. Lapidus has done several conjectures.

Bad news!!

Unfortunately, exactly as for smooth domains with symmetries, the second term can oscillate. This was conjectured by D.V.Vassiliev and we (JF+DV) could prove it for various examples.

Indeed very often examples of "fractals" are selfsimilar curves or in any case iterations are involved. It plays the role of symmetries as we will show on an example.

For these examples, the second term in (6) is not :

$$c_n M(\partial\Omega)\lambda^{d_i/2}$$

as hoped but is

$$c_n M(\partial\Omega)\lambda^{d_i/2}.p(\ln\lambda)$$

where p is a periodic function, positive, bounded and discontinuous.

An example Let s be a real fixed number such that

(13) $$1+\sqrt{2} < s < 3.$$

Let $\mathcal{Q}$ be the open set constructed the following way :

Q_o is the unit square;

one "sticks" in the middle of its for sides 4 squares Q_1 with side s^{-1};

on the $3 \times 4 = 12$ sides with length s^{-1} we stick $n_2 = 4 \times 3$ squares Q_2 with side $s^{-2}; \ldots.;$

at the k-th step we stick in the middle of the sides with length $s^{-(k-1)}$, $n_k = \frac{3}{4}3^k$ squares Q_k with side length $s^{-k} \ldots.$

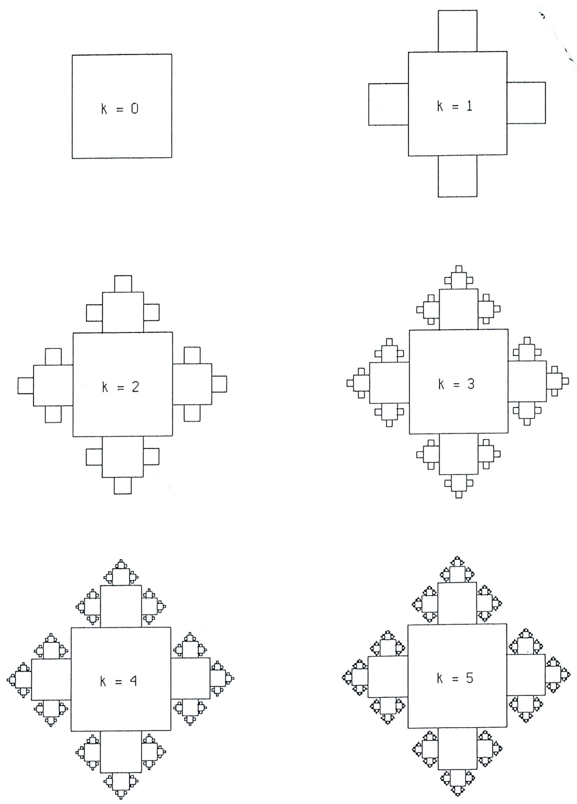


FIGURE 1

We denote by $\mathcal{Q}$ the union of all these squares for $k = 0, 1, 2, \ldots$. Note that $\mathcal{Q}$ is disconnected; moreover it follows from (2.1) that the squares do not overlap and that $\mathcal{Q}$ is with finite measure.

The open set $\mathcal{Q}$ is the union of all these squares. But it is not connected. Its boundary has an interior Bouligand-Minkowski dimension which is

$$d_i = (Ln3)/(Lns). \tag{14}$$

We derive from (13)($1+\sqrt{2} < s < 3$) and (14) that $1 < d_i < 2$.

For this open set we can prove

Proposition 1. *As* $\lambda \to +\infty$

$$N(\lambda, \mathcal{Q}) = W(\lambda, \mathcal{Q}) -$$

$$\frac{4}{3}(\frac{\lambda}{\pi^2})^{d_i/2} p_2[\frac{(Ln\lambda - 2Ln\pi)}{(2\ln s)}] + O(\lambda^{1/2})$$

with

(15)

$$p_2(y) := \sum_{k=-\infty}^{k=+\infty} 3^{k-y} \rho_2(s^{y-k}); \quad \rho_2(r) = \frac{\pi}{4} r^2 - \mathcal{N}_2(r),$$

where ρ_2 is the remainder term in Gauss estimate:

$$\mathcal{N}_2(r) = \#\{p, q) \in \mathbf{N}^* \times \mathbf{N}^* / p^2 + q^2 < r^2\}.$$

The function p_2 is well defined; it is positive, bounded, 1-periodic and left continuous; moreover the set of discontinuities is dense in $I\!R$.

We derive a connected set from $\mathcal{Q}$ by doing small cuts in the middle of each $\partial Q_k \cap \partial Q_{k-1}$. We denote by $\mathcal{O}$ this open connected set.

Theorem 1. *As* $\lambda \to +\infty$

$$-\frac{4}{3}(\lambda/\pi^2)^{d_i/2} p_2[(2\ln s)^{-1}(\ln\lambda - 2\ln\pi)] + O(\lambda^{1/2}) =$$

$$N(\lambda, \mathcal{Q}) - W(\lambda, \mathcal{Q}) \leq N(\lambda, \mathcal{O}) - W(\lambda, \mathcal{O}) \leq$$

$$-\frac{4}{3}(\lambda/\pi^2)^{d_i/2} p_2[(2\ln s)^{-1}(\ln\lambda - 2\ln\pi) + o(1)] + o(\lambda^{d_i/2}).$$

Remark The p_2 function appears naturally.

Sketch of the proof Since $\mathcal{Q}$ is the union of disjoint cubes

$$N(\lambda, \mathcal{Q}) = \sum_{k=0}^{\infty} n_k N(\lambda, Q_k). \tag{16}$$

$$\Rightarrow \; N(\lambda, \mathcal{Q}) - W(\lambda, \mathcal{Q}) = \sum_{k=0}^{\infty} n_k [N(\lambda, Q_k) - W(\lambda, Q_k)]$$

$$= \sum_{k=-\infty}^{\infty} n_k [N(\lambda, Q_k) - W(\lambda, Q_k)] -$$

$$\sum_{k=-\infty}^{-1} n_k [N(\lambda, Q_k) - W(\lambda, Q_k)]$$

Set

$$N(\lambda, \mathcal{Q}) - W(\lambda, \mathcal{Q}) = A + B \tag{17}$$

where

$$A := \frac{4}{3} \sum_{k=-\infty}^{\infty} 3^k \rho_2(\lambda^{1/2} s^{-k}/\pi)$$

$$B := -\rho_2(\lambda^{1/2}/\pi) + \frac{4}{3} \sum_{k=-\infty}^{0} 3^k \rho_2(\lambda^{1/2} s^{-k}/\pi).$$

We prove now that B is small compared to A.

Define $y \in I\!\!R$ by

(18) $\quad s^y = \sqrt{\lambda}/\pi,$ i.e $y = 2(Lns)^{-1}.\left[Ln\lambda - 2Ln\pi\right].$

$\Rightarrow\ 3^y = (\sqrt{\lambda}/\pi)^{d_i}$, hence

$$\sum_{k=-\infty}^{-1} 3^{k-y}\left[\mathcal{N}_2(s^{k-y}) - \pi s^{2(y-k)}/4\right] \simeq O(\sqrt{\lambda}),\ \lambda \to +\infty$$

so

(19) $$|B| \leq \gamma\lambda^{1/2}.$$

The definition of A exhibit the periodicity. Combining (17) to (19), we derive Proposition 1.

Analogously for $Z(t)$, with M.Levitin and D.Vassiliev we computed oscillating terms for the snowflake. (computing of the Heat content)

IV.7 INVERSE PROBLEME

We consider now the inverse problem knowing the spectral function, is it possible to deduce d or d_i?

This is of course important since it can help for example in the detection of cracks...

We use the partition function and our result ([FV1-2]) is closed to the one of *Brossard;Carmona*)

Theorem 2. *If* $\Omega \subset \mathbb{R}^n$ *is a bounded domain its interior Bouligand-Minkowski dimension* d_i *is such that:*

$$d_i \geq -2 \liminf \frac{Ln[|\Omega|(4\pi t)^{-n/2} - Z(t,\Omega)]}{Lnt}. \tag{20}$$

If Ω *is bounded in* $\mathbb{R}^2$ *and if* $\partial\Omega$ *has only a finite number of connected components then* (20) *is valid with an equality.*

These two conditions are necessary for having equality. Its is possible to construct counterexamples by use of the following remark:

If one extracts a sequence of points of the open set, the spectrum does not change but the Bouligand-Minkowski dimension of the boundary does.

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