Universal Bounds for Traces of the Dirichlet Laplace Operator

Leander Geisinger

Universität Stuttgart
leander.geisinger@mathematik.uni-stuttgart.de

Queen Dido Conference
2010

Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open set. Consider $-\Delta$ on Ω subject to Dirichlet boundary conditions with discrete eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$.
Define the *trace of the heat kernel*

$$Z(t) \;=\; \mathrm{Tr}\, e^{+\Delta t} \;=\; \sum_{k\in\mathbb{N}} \exp\left(-\lambda_k\, t\right)\,, \quad t > 0\,.$$

We have two fundamental results:

M. Kac, 1951

$$Z(t) \;\leq\; \frac{|\Omega|}{4\pi t} \quad \forall\, t > 0$$

Asymptotic expansion

$$Z(t) \;=\; \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{4\sqrt{4\pi t}} + O\left(1\right) \quad \text{as} \quad t \to 0+$$

Let $\Omega \subset \mathbb{R}^2$ be an open set. Consider $-\Delta$ on Ω subject to Dirichlet boundary conditions with discrete eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$.
Define the *trace of the heat kernel*

$$Z(t) \; = \; \mathrm{Tr}\, e^{+\Delta t} \; = \sum_{k \in \mathbb{N}} \exp\left(-\lambda_k\, t\right) , \quad t > 0 \, .$$

We have two fundamental results:

M. KAC, 1951

$$Z(t) \; \leq \; \frac{|\Omega|}{4\pi t} \quad \forall\, t > 0$$

ASYMPTOTIC EXPANSION

$$Z(t) \; = \; \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{4\sqrt{4\pi t}} + O\left(1\right) \quad \text{as} \quad t \to 0+$$

Auxiliary results and notation

As a lower bound on the ground state λ_1 one can always use

Rayleigh, Faber & Krahn

$$\tilde{\lambda} = \frac{\pi\, j_{0,1}^2}{|\Omega|} \leq \lambda_1 , \qquad (j_{0,1} : \text{first zero of the Bessel-function } J_0) .$$

Assume $a > 0$ and define

$$\hat{\Gamma}(a,s) = \frac{1}{\Gamma(a)} \int_s^\infty e^{-t}\, t^{a-1}\, dt .$$

Then we have the following asymptotic properties.

$$\hat{\Gamma}(a,s) = 1 - \frac{s^a}{a\,\Gamma(a)} + O\left(s^{a+1}\right) \quad \text{as} \quad s \to 0+$$

$$\hat{\Gamma}(a,s) = \frac{1}{\Gamma(a)} \left(s^{a-1} + O\left(s^{a-2}\right) \right) \exp(-s) \quad \text{as} \quad s \to \infty$$

Here we consider the case of an arbitrary open set $\Omega \subset \mathbb{R}^2$ with finite area and get the following universal bound on $Z(t)$.

THEOREM

For $\lambda \in [\tilde{\lambda}, \lambda_1]$ and all $t > 0$ the bound

$$Z(t) \;\leq\; \frac{|\Omega|}{4\pi t}\,\hat{\Gamma}\left(\frac{9}{2}, \lambda t\right) \;-\; (R(t,\lambda))_+$$

holds with a remainder term

$$R(t) \;=\; \frac{\sqrt{|\Omega|}}{\sqrt{4\pi t}}\,\frac{32}{35\sqrt{\pi}}\,\hat{\Gamma}\left(4, \lambda t\right) - \frac{\sqrt{4\pi t}}{\sqrt{|\Omega|}}\,\frac{\pi^{\frac{3}{2}}}{105}\,\hat{\Gamma}\left(6, \lambda t\right)\;.$$

With $\lambda = \tilde{\lambda}$ the bound depends only on t and $|\Omega|$.

We use upper bounds on the eigenvalue means

$$R_\sigma(\Lambda) \;=\; \mathrm{Tr}\,(-\Delta-\Lambda)_-^\sigma \;=\; \sum_k (\Lambda-\lambda_k)_+^\sigma\,.$$

BEREZIN, 1972

For $\sigma \geq 1$ and all $\Lambda > 0$

$$R_\sigma(\Lambda) \;\leq\; L_{\sigma,2}^{cl}\,|\Omega|\,\Lambda^{\sigma+1}\,.$$

The eigenvalue means are connected with the heat kernel via the Laplace transformation $\mathcal{L}[f](t) = \int_0^\infty e^{-\Lambda t} f(\Lambda)\,d\Lambda$:

$$Z(t) \;=\; \frac{t^{\sigma+1}}{\Gamma(\sigma+1)}\,\mathcal{L}[R_\sigma](t)\,, \quad \sigma \geq 1\,.$$

In this way the Berezin inequality implies Kac' inequality.

To state a refined bound choose a coordinate system $(x_1, x_2) \subset \mathbb{R}^2$. Define

$$l(x_1) = |\{x_2 : (x_1, x_2) \in \Omega\}| \quad \text{and} \quad m(\tau) = |\{x_1 : l(x_1) > \tau\}| \,.$$

Then we have $\int_0^\infty m(\tau) d\tau = |\Omega|$.

PROPOSITION

For $\sigma \geq 5/2$ *and all* $\Lambda > 0$

$$R_\sigma(\Lambda) \;\leq\; L^{cl}_{\sigma,2} \int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m(\tau)\, d\tau \;\Lambda^{\sigma+1} \,.$$

From this result one can deduce improvements of the Kac inequality via the Laplace transformation. But these improvements still depend on $m(\tau)$.

J. M. LUTTINGER, 1973

For all $t > 0$

$$Z(t,\Omega) \leq Z(t,B)\,, \tag{1}$$

where $B \subset \mathbb{R}^2$ is the circle with $|B| = |\Omega|$.

Conclusion:

- Calculate $m(\tau)$ for the circle B.
- Use the improved Berezin inequality to estimate R_σ for B.
- Apply the Laplace transformation and use (1) to deduce universal bounds on $Z(t)$.

J. M. Luttinger, 1973

For all $t > 0$

$$Z(t,\Omega) \;\leq\; Z(t,B)\,, \tag{1}$$

where $B \subset \mathbb{R}^2$ is the circle with $|B| = |\Omega|$.

Conclusion:

- Calculate $m(\tau)$ for the circle B.
- Use the improved Berezin inequality to estimate R_σ for B.
- Apply the Laplace transformation and use (1) to deduce universal bounds on $Z(t)$.

- Sharp upper bounds on $Z(t)$ and $R_\sigma(\Lambda)$ in domains with infinite volume as long as

$$\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m(\tau)\, d\tau \;<\; \infty$$

for all $\Lambda > 0$.

- Universal upper bounds on $R_\sigma(\Lambda)$ for $\sigma \geq 3/2$ with order-sharp correction terms.

- Improved upper bounds with correction terms depending on properties of the boundary.

- Corollary: For all $t > 0$ we have

$$Z(t) \;\leq\; \frac{|\Omega|}{4\pi t} \exp\left(-\frac{t}{|\Omega|}\right).$$

Further applications

- Sharp upper bounds on $Z(t)$ and $R_\sigma(\Lambda)$ in domains with infinite volume as long as
$$\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m(\tau)\, d\tau < \infty$$
for all $\Lambda > 0$.

- Universal upper bounds on $R_\sigma(\Lambda)$ for $\sigma \geq 3/2$ with order-sharp correction terms.

- Improved upper bounds with correction terms depending on properties of the boundary.

- Corollary: For all $t > 0$ we have
$$Z(t) \leq \frac{|\Omega|}{4\pi t} \exp\left(-\frac{t}{|\Omega|}\right).$$

- Sharp upper bounds on $Z(t)$ and $R_\sigma(\Lambda)$ in domains with infinite volume as long as

$$\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m(\tau)\, d\tau < \infty$$

for all $\Lambda > 0$.

- Universal upper bounds on $R_\sigma(\Lambda)$ for $\sigma \geq 3/2$ with order-sharp correction terms.

- Improved upper bounds with correction terms depending on properties of the boundary.

- Corollary: For all $t > 0$ we have

$$Z(t) \leq \frac{|\Omega|}{4\pi t} \exp\left(-\frac{t}{|\Omega|}\right).$$

- Sharp upper bounds on $Z(t)$ and $R_\sigma(\Lambda)$ in domains with infinite volume as long as

$$\int_{\frac{\pi}{\sqrt{\Lambda}}}^{\infty} m(\tau)\, d\tau \; < \; \infty$$

  for all $\Lambda > 0$.

- Universal upper bounds on $R_\sigma(\Lambda)$ for $\sigma \geq 3/2$ with order-sharp correction terms.

- Improved upper bounds with correction terms depending on properties of the boundary.

- Corollary: For all $t > 0$ we have

$$Z(t) \; \leq \; \frac{|\Omega|}{4\pi t} \exp\left(-\frac{t}{|\Omega|}\right) \; .$$