Torsional rigidity and Euclidean moment of inertia

Dinara Giniyatova

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Torsional rigidity for elastic beams

Let $\Omega \subset \mathbb{C}$ be a simply connected domain

$\partial\Omega$ is the boundary of Ω

$R(z, \Omega)$ is the conformal radius of Ω at the point $z = x + iy$

$\rho(z, \Omega)$ is the distance from z to $\partial\Omega$

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[Saint-Venant, 1847]

$$P(\Omega) = 2 \iint\limits_{\Omega} v(x, y)\, dxdy,$$

here $v(x, y)$ is the solution of the Dirichlet problem for the Poisson equation $\Delta v = -2\,, \quad v|_{\partial\Omega} = 0.$

Euclidean and conformal moments of inertia

$$I(\partial\Omega) = \iint_{\Omega} \rho^2(z,\Omega)\, dxdy$$

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$$\rho(z,\Omega) \le R(z,\Omega) \le 4\rho(z,\Omega) \tag{1}$$

[Avkhadiev,F.G. *Solution of the generalized Saint Venant problem* Mat. Sb., 1998, **189**(12), 3–12]

Theorem

For simply connected domains $\Omega \subset \mathbb{C}$

$$P(\Omega) \sim I(\partial\Omega) \sim I_c(\Omega)$$

rather, the following chain of inequalities is valid:

$$I(\partial\Omega) \leq I_c(\Omega) \leq P(\Omega) \leq 4I_c(\Omega) \leq 64I(\partial\Omega) \qquad (2)$$

For convex domains

$$P(\Omega) < 4I(\partial\Omega) \tag{3}$$

What about other domains?

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What about other domains?

Kovalev, L.V. (2002)

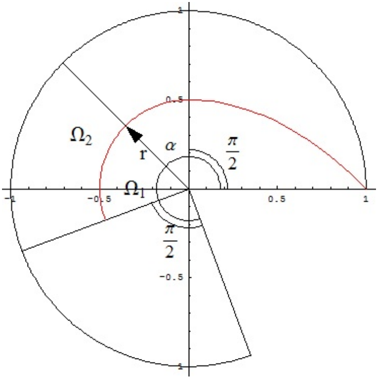
It is not true!

$$\frac{P(\Omega)}{I(\partial\Omega)} > 4$$

Theorem

There exists the simply connected domain $\Omega \subset \mathbb{C}$ *for which* $P(\Omega)/I(\partial\Omega) > 4.08$

Sector of radius a ($\alpha > \pi$)



$$I(\partial\Omega) = \frac{a^4}{360}(15(\alpha + \pi) - 64)$$

Torsional rigidity for an arbitrary sector

$$P(\Omega) =$$

$$a^4 \left\{ \frac{\pi\lambda}{2} - \frac{1}{\pi} \left[\psi\left(\frac{1}{2} + 2\lambda\right) - \psi\left(\frac{1}{2}\right) - \lambda\psi'\left(\frac{1}{2} + 2\lambda\right) \right] \right\}$$

$$\alpha = 2\pi\lambda, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

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$$\alpha > \alpha^* \approx 1,81948\pi (\approx 327,5^\circ)$$

Thank you for your attention!