Existence results for critical semi-linear equations on Heisenberg group domains.

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Let

$$(P)\begin{cases} -\Delta_H u = K u^3 & on \quad \Omega \\ u > 0 & on \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

where Δ_H is the sublaplacian of $\mathbb{H}^1$, K is a C^3 positive function and Ω is a bounded and smooth domain in $\mathbb{H}^1$. Using a version of Morse lemma at infinity, we give necessary conditions on K to insure the existence of solutions of (P).

Background and motivations

After the work of D.Jerison and J.M.Lee (1987) on the Yamabe problem on CR manifolds, different methods established to prove existence and nonexistence results for semilinear equations for the Kohn Spencer Laplacian :

- variational methods

- N.Garofallo and E.Lanconelli (1992)
(They proved for Yamabe type problem ($K = 1$ in (P))a nonexistence result in the case where Ω is a H-starshaped domain and they gave the first example of solution of (P) for a non-contractible domain $\Omega = \{(x, y, t)/r < |x|^2 + |y|^2 < R, 0 < t < T\}$.)
- I.Birendelli and A.Cutri ; S. Biagini ; G.Citti (1995) .

- super and sub-solutions methods

- G.Lu and J.Wei(1997) ;
- L. Brandolini, M. Rigoli and A.G.Setti (1998) ;
- F.Uguzzoni (1999).

- blow-up techniques

- I.Birendelli (1997) ;
- I.Birendelli, I.Capuzzo Dolcetta and A.Cutri (1998).....

- mean value formulas

- E.Lanconelli and F.Uguzzoni (1998) ;
- F.Uguzzoni (1999).

- moving plan methods

- I.Birendelli and J.Prajapat.

G.Citti and F.Uguzzoni in (2001) studied the CR version of a famous theorem due to A.Bahri and J.M.Coron and proved an existence result for Yamabe type problem on euclidian domains which have a nontrivial homology group (with $\mathbb{Z}_2$-coefficients).

In our case, we are interested by an analogous problem to the one studied by M.Ben Ayed and M.Hammami in the Euclidean case. We will use the same techniques presented by N.Gamara, where such problem was studied for spherical CR manifolds without boundary and known to be "the Webster scalar curvature problem".

We have to mention that for a Heisenberg group domain a crucial point is played by the outward normal of the boundary, where delicate expansions for the function K, the Green's function associated to the laplacian and its regular part near the boundary, are essential tools to investigate a solution for such problem.

Hypotheses :

H.1

K has only nondegenerate critical points $y_1, y_2, ..., y_m$ such that :

$$-\frac{\Delta_H K(y_i)}{3K(y_i)} + \frac{2}{c_q} H(y_i, y_i) \neq 0\,,\ i = 1, ..., m, \tag{1}$$

and $y_1, y_2, ..., y_{m_1}$ are all the critical points of K with

$$-\frac{\Delta_H K(y_i)}{3K(y_i)} + \frac{2}{c_q} H(y_i, y_i) > 0\,,\ i = 1, ..., m_1. \tag{2}$$

H.2

For each $\xi \in \partial\Omega$

$$\frac{\partial K(\xi)}{\partial \overrightarrow{\nu}} < 0\,, \tag{3}$$

where $\overrightarrow{\nu}$ is the intrinsic outer unit normal to Ω.

We assume that Ω is a domain without characteristic point satisfying the uniform exterior ball property. It yields that for all φ in $C(\partial\Omega)$, the Dirichlet problem ($\Delta_H u = 0$ in Ω , $u = \varphi$ in $\partial\Omega$) has a classical solution $u \in H(\Omega) \cap C(\overline{\Omega})$.
In particular Ω has a Green's function defined by

$$G(\xi, \xi^{'}) = \frac{c_q}{d^2(\xi, \xi')} - H(\xi, \xi^{'}), \ \forall \xi, \xi^{'} \in \Omega, \tag{4}$$

where H is the regular part of G.

$$\begin{cases} \Delta_H \quad H(\xi, .) & = & 0 & \text{in } \Omega \\ \qquad\ \ H(\xi, .) & = & \Gamma(\xi, \xi') & \text{on } \partial\Omega. \end{cases} \tag{5}$$

For $r \in \mathbb{N}^*$ and $\xi_i \in \Omega$ for all $1 \leq i \leq r$, such that $\xi_i \neq \xi_j \ \ \forall i \neq j$; we denote by $M(\xi) = (m_{i\,j})_{1 \leq i,j \leq r}$ the matrix defined by

$$\begin{cases} m_{i\,i} & = & -\frac{\Delta_H K(\xi_i)}{3K(\xi_i)^2} + \frac{2H(\xi_i,\xi_i)}{c_q K(\xi_i)} \\ \\ m_{i\,j} & = & -\frac{2G(\xi_i,\xi_j)}{c_q \left(K(\xi_i)K(\xi_j)\right)^{\frac{1}{2}}}\ , \qquad 1 \leq i \neq j \leq r \end{cases} \tag{6}$$

H.3

For $\tau_s = (i_1, i_2, ..., i_s)$ any s-tuple of $(1, ..., m_1)$, $i_j \neq i_k$ for $j \neq k$, the matrix $M(\tau_s) = M(y_{i_1}, ..., y_{i_s})$ defined by (6) is nondegenerate.

Existence result

Theorem

Under assumptions H.1, H.2 and H.3, if

$$\sum_{s=1}^{m_1} \sum_{\tau_s=(i_1,\ldots,i_s), M(\tau_s)>0} (-1)^{4s-1-\sum_{j=1}^{s} k_{i_j}} \neq 1 \,, \qquad (7)$$

where k_{i_j} denotes the index of the critical point y_{i_j} with respect to K, then (P) has a solution.

Technical results for the Green's function and its regular part

Lemma 1

For each $\xi \in \Omega$, near the boundary of Ω, let $\overrightarrow{\nu}_\xi = \overrightarrow{\nu}$ be the intrinsic outer unit normal to $\partial\Omega$ at ξ, and H the regular part of the Green's function at this point, we have the following estimates

$$\begin{aligned}
&i/ \quad H(\xi,\xi) = \frac{c_q}{(2d_\xi)^2} + o(d_\xi^{-2})\\
&ii/ \quad H(\eta,\xi) \leq c(\max(d_\xi, d_\eta))^{-2}\\
&iii/ \quad |\frac{\partial H}{\partial \eta}|(\eta,\xi) \leq \frac{c}{d_\eta} H(\eta,\xi)\\
&iv/ \quad |\frac{\partial H}{\partial \overrightarrow{\nu}}|(\xi,\xi) \leq \frac{c_q}{2d_\xi^3} + o(\frac{1}{d_\xi^3})\\
&v/ \quad H(\eta,\eta) \geq \ c.
\end{aligned}$$

Lemma 2

For each ξ_1 and ξ_2 in Ω, near the boundary, we denote d_i the distance of ξ_i to $\partial\Omega$. Let $\overrightarrow{\nu}_1$ be the intrinsic outer unit normal to $\partial\Omega$ at ξ_1 and H the regular part of the Green's function. If $\frac{d_1}{d_2}$, $\frac{d_2}{d_1}$ and $\frac{d(\xi_1,\xi_2)}{d_1}$ are bounded, then

$$(\frac{\partial H}{\partial \overrightarrow{\nu}_1})(\xi_1,\xi_2) \ > \ 0.$$

Proposition 1(Maximum principle)

Let Ω be subdomain of $\mathbb{H}^1$ without characteristic point satisfying an interior ball condition at $\xi_0 \in \partial\Omega$. Suppose that
(i) $\Delta_H u \geq 0$ in Ω,
(ii) u is continuous at ξ_0 ;
(iii) $u(\xi_0) > u(\xi)$ for all $\xi \in \Omega$;
Then

$$\frac{\partial u}{\partial \overrightarrow{\nu}}(\xi_0) > 0$$

where $\overrightarrow{\nu}$ is the unit outer normal to $\partial\Omega$ at ξ_0.

Lemma 3

Let $(\xi_1, \xi_2) \in \Omega^2$ be such that $d_1 \leq d_2$ and $c_2 d_2 \leq d(\xi_1, \xi_2)$, where c_2 is a fixed constant. If d_1 is small enough, then

$$(\frac{\partial G}{\partial \overrightarrow{\nu}_1})(\xi_1, \xi_2) \leq 0.$$

There exists a natural orthogonal projection

$$\begin{array}{rcl} P : S_0^{1,2}(\mathbb{H}^n) & \longrightarrow & S_0^{1,2}(\Omega) \\ \phi & \longmapsto & P\phi \end{array}$$

defined by the equation :

$$P\phi = \phi - h \Longleftrightarrow \left\{ \begin{array}{rcl} \Delta_H P\phi & = & \Delta_H \phi \text{ in } \Omega \\ P\phi & = & 0 \quad\quad \text{on } \partial\Omega. \end{array} \right.$$

Lemma 4

For the function h_i, such that $P\delta_{(a_i,\lambda_i)} = \delta_{(a_i,\lambda_i)} - h_i$ and $H(\xi, .)$ the regular part of the Green's function, we have the following estimates for h_i :

$$
\begin{aligned}
i/ \quad & h_i(\xi) = \frac{H(a_i,\xi)}{c_q\lambda_i} + O(\frac{1}{\lambda_i^3 d_i^4}) \\
ii/ \quad & \frac{\partial h_i(\xi)}{\partial a_i} = \frac{1}{c_q\lambda_i}\frac{\partial H(a_i,\xi)}{\partial a_i} + O(\frac{1}{\lambda_i^3 d_i^5}) \\
iii/ \quad & \lambda_i\frac{\partial h_i(\xi)}{\partial \lambda_i} = -\frac{H(a_i,\xi)}{c_q\lambda_i} + O(\frac{1}{\lambda_i^3 d_i^4}) \\
iv/ \quad & |h_i|_{L^4(\Omega)} = O(\frac{1}{\lambda_i d_i}) \\
v/ \quad & |\frac{1}{\lambda_i}\frac{\partial h_i}{\partial a_i}|_{L^4(\Omega)} = O(\frac{1}{\lambda_i^2 d_i^2}) \\
vi/ \quad & |\lambda_i\frac{\partial h_i}{\partial \lambda_i}|_{L^4(\Omega)} = O(\frac{1}{\lambda_i d_i}).
\end{aligned}
$$

Proof's idea of the theorem

- By contradiction : we assume that $S_{(P)} = \emptyset$.

Let $V_{\varepsilon_0}(\Sigma^+) = \{u \in \Sigma \ / \ J(u)^2 \left|u^-\right|_{L^4} < \varepsilon_0\}$.

- J and $\partial J \Rightarrow$ Estimations of G and H near $\partial\Omega \Longrightarrow W = \sum_{comb.conv.} W_i$.
- $V_{\varepsilon_0}(\Sigma^+)$ is contractible and $\chi(V_{\varepsilon_0}(\Sigma^+)) = 1$.
- $\chi(V_{\varepsilon_0}(\Sigma^+)) = \sum_{s=1}^{m_1} \sum_{\tau_s=(i_1,\dots,i_s), M(\tau_s)>0} (-1)^{4s-1-\sum_{j=1}^{s} k_{i_j}}$.
- Thus a contradiction with $\chi(V_{\varepsilon_0}(\Sigma^+)) \neq 1$.
- We deduce $S_{(P)} \neq \emptyset$.

References :

- 1- A.Bahri : Critical points at Infinity in some Variational Problems, Pitman Research Notes in Mathematics Series, (1989) , MR 91h :58022, Zbl 676.58021.
- 2- A.Bahri - Y.Y.Li - O.Rey : On a varational problem with lack of compactness : The topological effect of the critical points at infinity, Calculus of Variation and Partial Diff. Equa., 3 (1995), 67-94.
- 3- M.Ben Ayed - M.Hammami : On a variational problem involving critical sobolev growth in dimention four, Advances in Differential Equations, volume 9 3-4 (2004), 415-446.
- 4- N.Gamara : The Prescribed Scalar Curvature on a 3-Dimensional CR Manifold, Advanced Nonlinear Studies 2 (2002), 193-235.
- 5- O. Rey : A multiplicity result for a variational problem with lack of compatness, Nonlinear Analysis, Theory, Methods and Applications, Vol. 13, No. 10, (1989), 1241-1249.
- 6- F.Uguzzoni - E.Lanconelli : On the Poisson Kernel for the Kohn Laplacian, Rend. Math. Appl. 17 (1997), 659-677.

Thanks for your attention.