Some shape optimization problems with a polygonal solution

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Introduction

In this talk, we consider the class

$$\mathcal{K} = \{K \subset \mathbb{R}^2 ; K \text{ convex set}\}$$

with various other constraints.

We are interested in minimization (or maximization) problems on various subsets of the class $\mathcal{K}$ whose solutions are singular, actually polygons.

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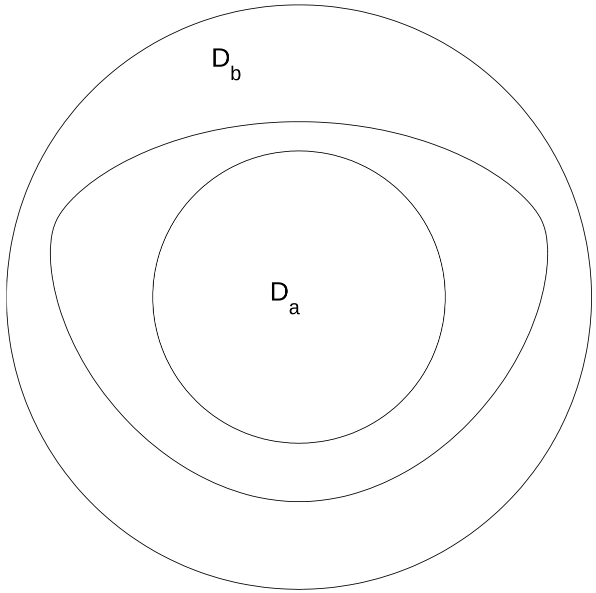
More precisely, we want to answer the following Question 1: Let C be given in $\mathcal{A}$, what is the farthest convex set K_C such that $d(K_C, C) = \max_{K\in\mathcal{A}} d(K, C)$.

Example 2: a geometric problem

Let D_a and D_b be the disks, centered at O of radius a, b. Let $\mathcal{B}$ be the class $\mathcal{B} = \{D_a \subset K \subset D_b, K$ convex$\}$.
Let $\alpha > 0$ be given. We want to minimize

$$J_\alpha(K) = \alpha|K| - P(K)\,.$$

Question 2: What is the solution for any α?



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The problem $\max\{\lambda_1(\Omega), \Omega$ convex $, |\Omega| = V_0, P(\Omega) \geq P_0\}$ is well-posed.
Question 3: What is the maximizer of the first eigenvalue among convex domains with perimeter and area constraints?

Example 4: the Mahler conjecture

Let K be a symmetric convex body and K° its polar body: $K^\circ = \{\xi; |x.\xi| < 1, \forall x \in K\}$. The Mahler volume is $M(K) = |K||K^\circ|$. It is invariant by affine transformation.

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Question 4: What are the minimizers of the Mahler volume?

In 2D, it is the square (Mahler). In higher dimensions, it is a famous conjecture: it should be the cube (Difficult question: see T. Tao's blog).

A geometric approach

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For domains, convexity or concavity properties are known as Brunn-Minkowski inequalities. For example, in the plane, $|K|^{1/2}$ or $\lambda_1(\Omega)^{-1/2}$ are strictly concave:

$$|(1-t)K_0 + tK_1|^{1/2} \geq (1-t)|K_0|^{1/2} + t|K_1|^{1/2}$$

with equality iff K_0, K_1 are homothetic.

Indecomposability

Definition K is indecomposable (in $\mathcal{M}$) if

$$K = (1-t)K_0 + tK_1 \text{ (with } K_0, K_1 \in \mathcal{M}\text{)}$$

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Example Minimizers of the logarithm capacity in $\mathbb{R}^2$ among convex sets of given perimeter are triangles or segments.

The support function(1)

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The support function h_K of K is defined by:

$$h_K(\theta) := \max\{x \cdot e^{i\theta} : x \in K\}\,.$$

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The Steiner point $s(K)$ of the convex set is defined by:

$$s(K) = \frac{1}{\pi}\int_0^{2\pi} h_K(\theta) e^{i\theta}\, d\theta\,.$$

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The polygons are also well characterized

$$K \text{ is a polygon} \iff h_K'' + h_K = \sum_{j=1}^{n} a_j \delta_{\theta_j}$$

where $a_1, a_2, \ldots, a_n$ and $\theta_1, \theta_2, \ldots, \theta_n$ denote the lengths of the sides and the angles of the corresponding outer normals.

Support function and distances

The Hausdorff distance can be defined using the support functions:

$$d_H(K, L) = \|h_K - h_L\|_\infty.$$

We can also define a L^2 distance (Mc Clure and Vitale) by

$$d_2(K, L) := \left(\int_0^{2\pi} |h_K - h_L|^2 \, d\theta \right)^{1/2}.$$

The farthest convex set (L^2)

(Joint work with Evans Harrell)

Theorem Let J be a functional defined by

$$J(K) := \int_0^{2\pi} a\, h_K^2 + b\, {h'_K}^2 + c\, h_K + d\, h'_K \, d\theta$$

(example the L^2 distance: $J(K) = \int_0^{2\pi} (h_K - h_C)^2 \, d\theta$). Then every local maximizer of the functional J within the class $\mathcal{A}$ is either a segment or a triangle.

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Corollary The farthest convex set for the L^2 distance is either a segment or a triangle.

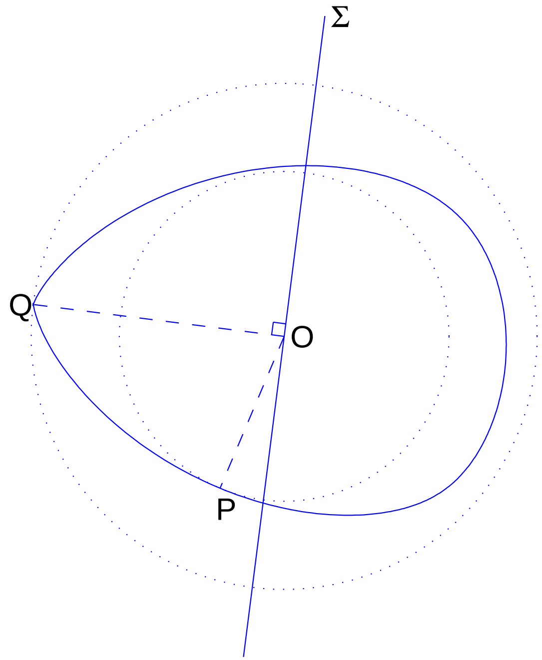
Actually, we can prove that it is a segment

The farthest convex set (Hausdorff)

Theorem [farthest convex set for Hausdorff distance]
If C is a given convex set in the class $\mathcal{A}$, then the convex set K_C for which

$$d_H(C, K_C) = \max\{d_H(C, K) : K \in \mathcal{A}\}$$

is a segment.



Ingredient: a geometric inequality

Theorem Let K be any plane convex set with its Steiner point at the origin. Then

$$\max h_K \leq \frac{P(K)}{4} \leq \min h_K + \max h_K,$$

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The first inequality is due to P. Mc Mullen. It implies that the diameter of $\mathcal{A}$ is less than $\pi/2$.

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Question: Does it imply that it is a triangle? If yes, it can be proved that the maximizer is an isosceles triangle.

Analytic approach

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Lemma[T. Lachand-Robert,M. Peletier, J. Lamboley, A. Novruzi] If suppt$(h'' + h)$ has at least 3 points in $(0, \varepsilon)$, there exists v compactly supported in $(0, \varepsilon)$ such that $h + tv$ is the support function of a convex set.

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Consequence: If $J(K) = j(h_K)$ is strictly "locally concave" in h, the minimizers have to be polygons.

A geometric example

(Work with Chiara Bianchini)

We recall that we want to minimize $J_\alpha(K) = \alpha|K| - P(K)$ in the class $\mathcal{B} = \{D_a \subset K \subset D_b, K$ convex$\}$.

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Using the support function, we have

$$J_\alpha(K) = \alpha \int_0^{2\pi} h_K^2 - {h'_K}^2 - \int_0^{2\pi} h_K$$

Let $h(= h_K)$ be a minimizer and assume that the support of $h'' + h$ has at least 3 points in some interval $(0, \varepsilon)$.
According to the previous Lemma, there exists v compactly supported in $(0, \varepsilon)$ such that $h + tv$ is the support function of a convex set.

A geometric example (2)

Then the second derivative J''_α must be non negative at v:

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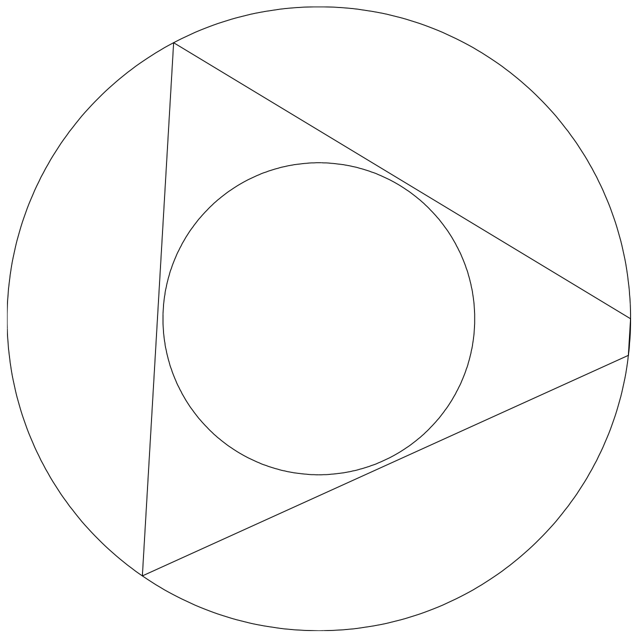
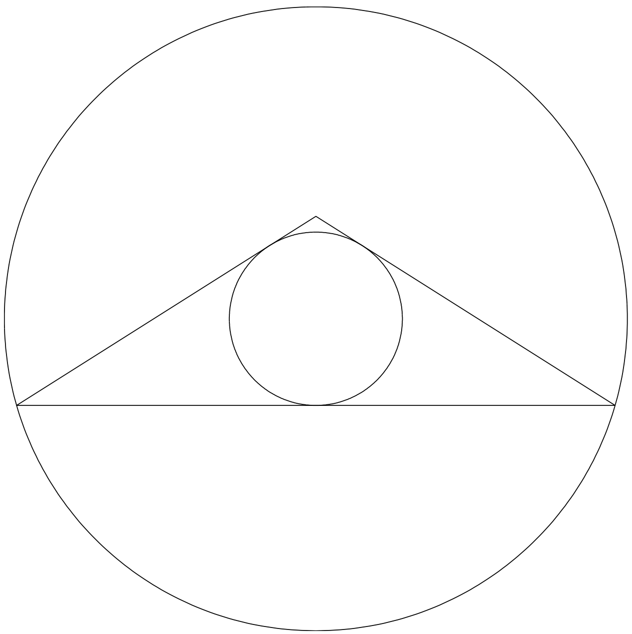
Consequence: Any (local) minimizer is a polygon inside the ring

A geometric example (3)

What are the minimizer(s)? Of course, it depends on the parameters a, b, α.



$a = 1, b = 2, \alpha = 0.33$



$a = 1, b = 3, \alpha = 1.5$

The Mahler conjecture (1)

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We recall that we want to minimize the Mahler volume $M(K)=|K||K^\circ|$ where K° is the polar body of K.

In terms of the support function, the (2-D) Mahler volume can be expressed as

$$M(h_K)=\left(\int_0^{2\pi} h_K^2 - {h'_K}^2\right)\left(\int_0^{2\pi}\frac{1}{2h_K^2}\right)$$

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Aim: apply this technique in higher dimensions to make progress in the general conjecture.