Delaunay Surfaces in $\mathbb{S}^3$

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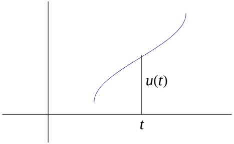
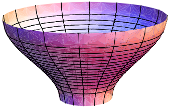
Part 1: Delaunay Surfaces in $\mathbb{R}^3$

CMC surfaces of revolution in $\mathbb{R}^3$





Easy computation

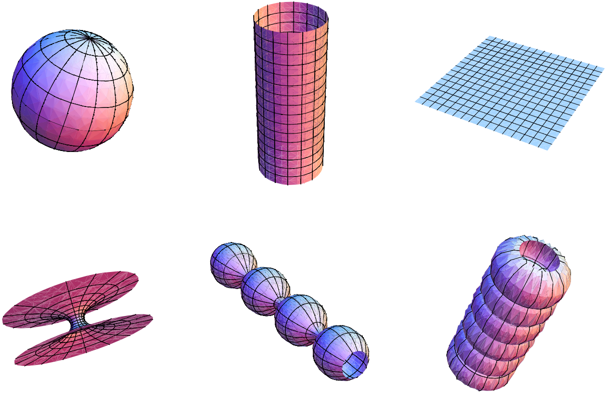
- Plane curve $t \mapsto (t, u(t))$
- Rotate around z-axis

$$X(t,\theta) = (t\cos\theta, t\sin\theta, u(t))$$

- Integrate mean curvature equation

$$2H = \frac{1}{t}\left(\frac{tu'}{\sqrt{1+u'^2}}\right)' \quad \Rightarrow \quad \boxed{\frac{u'}{\sqrt{1+u'^2}} = Ht + \frac{c}{t}}$$

Solutions



Other things to know

Theorem (Delaunay 1841)

The generating curves of CMC surfaces of revolution are roulettes of conic sections.

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Applications

- Used to build other CMC surfaces in $\mathbb{R}^3$ (K-noids, Bubbletons)
- Arise in Capillarity (modeling soap films, Double Bubble problem)

Part 2: Delaunay Surfaces in $\mathbb{S}^3$

Symmetry in $\mathbb{S}^3$

Definition

$M \subset \mathbb{S}^3$ is *symmetric, with respect to a circle* C, if

$$\Psi(M) = M$$

for each conformal $\Psi : \mathbb{S}^3 \to \mathbb{S}^3$ that fixes C pointwise.

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Lemma

$M \subset \mathbb{S}^3$ is symmetric iff there is a rotation O such that $\pi \circ O(M)$ is rotationally symmetric in $\mathbb{R}^3$. Here

$$\pi : \mathbb{S}^3 \setminus \{e_4\} \to \mathbb{R}^3; (x_1, x_2, x_3, x_4) \mapsto \frac{(x_1, x_2, x_3)}{1 - x_4}$$

is stereographic projection.

Stereographic projection, reflection, and inversion

Corollary

$M \subset \mathbb{S}^3$ is symmetric with respect to a circle C iff there

$$\Psi_Q(M) = M$$

for each 2-sphere Q, $C \subset Q \subset \mathbb{S}^3$.

A convenient parametrization

Theorem

$M \subset \mathbb{S}^3$ immersed and symmetric. Then there is a rotation O such that $\pi \circ O(M)$ admits the (local) parametrization

$$\boxed{X(\theta, \phi) = (R(\theta, \phi)\cos\theta, R(\theta, \phi)\sin\theta, h + r(\theta)\sin\phi)}$$

$(\theta, \phi) \in (-\epsilon, \epsilon) \times [0, 2\pi)$, *for <u>some</u> $h \in \mathbb{R}$ and $\rho > 0$. Here*

$$R(\theta, \phi) := \sqrt{r(\theta)^2 + \rho^2} + r(\theta)\cos\phi.$$

Mean curvature calculation

Expression for the mean curvature

- In general

$$\boxed{H_{\pi^{-1}(X)} = \frac{1+|X|^2}{2} H_X + X \cdot N}$$

- For $X = (R\cos\theta, R\sin\theta, h + r\sin\phi)$,

$$H_{\pi^{-1}(X)} = \frac{c_0 + c_1 \cos\phi + c_2(\cos\phi)^2}{4rR^2(r'^2 + d^2)^{3/2}}$$

  where $d = \sqrt{\rho^2 + r^2}$ and c_0, c_1, c_2 depend on r, r' and r''.
- NOTE: $H_{\pi^{-1}(X)}$ and H_X are <u>even functions</u> of ϕ.

Important lemma

Lemma

If $M \subset \mathbb{S}^3$ *is a non-spherical symmetric surface, we can parametrize* M *with the "convenient" parameterization with* $(h, \rho) = (0, 1)$.

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Proof.

Suppose $h \neq 0$.

$$\begin{aligned} 0 &= \partial_\phi H_M \Big|_{\phi=0} \\ &= \left(X \cdot X_\phi H_{\pi(M)} + X \cdot N_\phi\right) \Big|_{\phi=0} \\ &= h \left(r H_{\pi(M)} \Big|_{\phi=0} + \sqrt{\frac{r^2 + \rho^2}{r'^2 + r^2 + \rho^2}} \right). \end{aligned}$$

Then $\boxed{(r^2 + \rho^2) r r'' + \rho^2 r'^2 + (r^2 + \rho^2)^2 = 0}$, which corresponds to a sphere. □

A few corollaries

Corollary

$M \subset \mathbb{S}^3$ *is a symmetric surface,*

$$H_M = \frac{\sqrt{1+r^2}\,\left(r(r^2+1)r'' - r'^2 + r^4 - 1\right)}{2r(1+r^2+r'^2)^{3/2}}.$$

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Corollary

$M \subset \mathbb{S}^3$ is a CMC $(H_M \equiv H)$ symmetric surface,

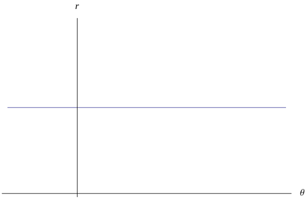
$$\frac{1}{\sqrt{1+r^2}\sqrt{1+r^2+r'^2}} + \frac{H}{1+r^2} = c$$

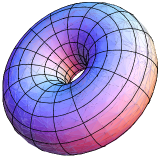
where c is constant.

Standard Tori

parameter values

$H = c - 1/4c$

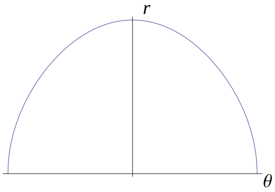


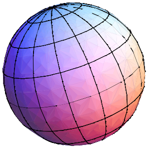


Spheres

parameter values

$H = c$

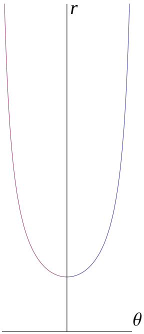


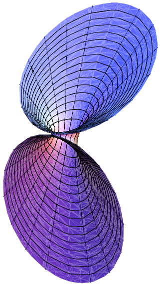


Catenoid types

parameter values

$H \neq 0, c = 0.$

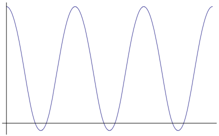
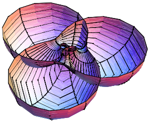
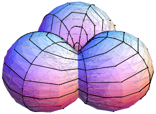




Unduloid types

parameter values

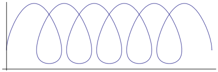
$c > 0$, $c - 1/4c < H < c$ and $c < 0$, $c < H < c - 1/4c$.

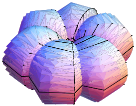




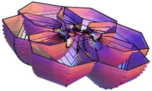
Nodoid types

parameter values

$H > 0$, $0 < c < H$ and $H < 0$, $H < c < 0$.







Period analysis

Period of Unduloids and Nodoids

$$T := \int_{r_{\min}}^{r_{\max}} \frac{2(c(r^2+1)-H)}{\sqrt{1+r^2}\sqrt{r^2-(c(r^2+1)-H))^2}} dr$$

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Proposition

M is an immersed torus iff

$$T \in 2\pi\mathbb{Q}.$$

For unduloid types, this immersion is an embedding if

$$T = 2\pi/m, \quad m \geq 2.$$

This happens only for $H \neq 0$. In this case, there is $c = c(m, H)$ corresponding to an embedded unduloid type with m bulges and necks.

Classification theorem

Theorem

- The complete symmetric CMC surfaces in $\mathbb{S}^3$ are: standard tori, spheres, catenoid types, unduloid types, and nodoid types.

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- There are (strictly immersed) catenoid, unduloid and nodoid type tori.
- For $H \neq 0$, there are embedded unduloid type tori.

Two interesting and motivating open problems

Sterling and Pinkall's Conjecture (1989)

Embedded CMC tori are symmetric.

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Lawson's Conjecture (1970)

The unique embedded minimal torus is the Clifford torus

$$\mathcal{C} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{S}^3 : x_1^2 + x_2^2 = 1/2 = x_3^2 + x_4^2 \right\}.$$

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Recall: Aleksandrov's Theorem (1956)

The round sphere is the unique compact embedded CMC surface in $\mathbb{R}^3$.

Rolling interpretation?

An observation:

Suppose $M \in \mathbb{S}^3$ is symmetric and $\pi(C)$ is the unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$. Then for each $\rho \geq 1$

$$\pi_\rho(\pi(M) \cap S_\rho)$$

is a plane curve parametrized via

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Conjecture

The above curve can also be obtained by trace of the focus of a conic section that rolls without slipping on the unit circle.