Complex Hessian equations on some compact Kähler manifolds

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Outline

1. Introduction : Kähler manifolds.
2. Theorem of extraction of a unique k-th root from a prescribed volume form : statement and reformulation.

Definition of a Kähler manifold :
M complex manifold of $dim_{\mathbb{C}}\, m$ (holomorphic change of charts)

$\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m}, \frac{\partial}{\partial z^{\overline{1}}}, \ldots, \frac{\partial}{\partial z^{\overline{m}}}$ basis of $T^{\mathbb{C}}M$.

$$J : T^{\mathbb{C}}M \to T^{\mathbb{C}}M$$

$$\frac{\partial}{\partial z^k} \mapsto i\frac{\partial}{\partial z^k}, \quad \frac{\partial}{\partial z^{\overline{k}}} \mapsto -i\frac{\partial}{\partial z^{\overline{k}}}$$

$\to J^2 = -Id$

$\to T^{1,0} := Vect(\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m}), \quad T^{0,1} := Vect(\frac{\partial}{\partial z^{\overline{1}}}, \ldots, \frac{\partial}{\partial z^{\overline{m}}})$

endowed with a metric g satisfying

- J-compatible : $g(JX, JY) = g(X, Y) \quad \forall X, Y \in TM$
- the associated $(1,1)$-form

$$\omega(X, Y) = g(JX, Y)$$

$$\omega = i\, g_{a\overline{b}} \quad dz^a \wedge dz^{\overline{b}} \quad g_{a\overline{b}} = g\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^{\overline{b}}}\right)$$

is closed. $\to \omega$ **Kähler form**.

For (M, J, g, ω) Kähler :

- $Mat_{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m}, \frac{\partial}{\partial z^{\overline{1}}}, \ldots, \frac{\partial}{\partial z^{\overline{m}}}} \quad g = \left[\begin{array}{c|c} 0 & H \\ \hline {}^tH & 0 \end{array}\right]$
  $H = (g_{a\overline{b}})_{1 \leq a,b \leq m}$ hermitian.
- $(g^{a\overline{b}})_{1 \leq a,b \leq m} := {}^t(g_{a\overline{b}})^{-1}_{1 \leq a,b \leq m}$
- $v_g = \frac{\omega^m}{m!} = i^m \, detg \quad dz^1 \wedge dz^{\overline{1}} \wedge \ldots \wedge dz^m \wedge dz^{\overline{m}}$

Let (M, J, g, ω) be a **compact connected** Kähler manifold of complex dimension $m \geq 3$.
Let us fix an integer $2 \leq k \leq m-1$.

If $\varphi : M \to \mathbb{R}$ C^∞,
(1,1)-form $\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi \qquad (\to [\tilde{\omega}] = [\omega]\ )$
the associated 2-tensor $\tilde{g}(X, Y) = \tilde{\omega}(X, JY) \quad \forall X, Y \in TM$
$(\to \tilde{g}_{a\bar{b}} = g_{a\bar{b}} + \partial_{a\bar{b}}\varphi)$

The sequilinear forms h and $\tilde{h}$ on $T^{1,0}$ defined as follows :

$$h(U, V) = g(U, \overline{V}) \quad \text{and} \quad \tilde{h}(U, V) = \tilde{g}(U, \overline{V}) \quad \forall U, V \in T^{1,0}$$

$\lambda(g^{-1}\tilde{g}) :=$ the vector of eigenvalues of $\tilde{h}$ with respect to the hermitian form h.
$\lambda(g^{-1}\tilde{g}) \in \mathbb{R}^m$.

$$\Gamma_k = \{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k \quad \sigma_j(\lambda) > 0\}$$

$\sigma_j :=$ j-th elementary symmetric function.
$\sigma_j(x_1, ..., x_m) = \sum_{1 \leq i_1 < ... < i_j \leq m} x_{i_1} ... x_{i_j}$

φ is called k-**admissible** if $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$.

Theorem (J. 2009)

*Let (M, J, g, ω) be a **compact connected** Kähler manifold of complex dimension $m \geq 3$ with **non-negative holomorphic bisectional curvature**, and let $f : M \to \mathbb{R}$ be a function of class C^∞ satisfying $\int_M e^f \omega^m = \binom{m}{k} \int_M \omega^m$. There exists a unique function $\varphi : M \to \mathbb{R}$ of class C^∞ such that :*

1. $\int_M \varphi\, \omega^m = 0$;
2. $\tilde{\omega}^k \wedge \omega^{m-k} = \frac{e^f}{\binom{m}{k}} \omega^m \quad (E_k)$.

Moreover the solution φ is k-admissible.

REMARKS

1. The normalisation condition $\int_M e^f \omega^m = \binom{m}{k} \int_M \omega^m$ is a necessary condition, indeed

$$\int_M \tilde{\omega}^k \wedge \omega^{m-k} = \int_M \omega^m$$

2. The case $k = m$ corresponds to Calabi-Yau : $(E_m) \quad \tilde{\omega}^m = e^f \omega^m$
3. The case $k = 1$ is trivial : $(E_1) \quad \tilde{\omega} \wedge \omega^{m-1} = \frac{e^f}{m} \omega^m$,

$$(E_1) \quad \triangle_g^{\mathbb{C}} \varphi = m - e^f \quad \text{avec} \quad (N) : \int_M (m - e^f)\omega^m = 0$$

4. φ is only k-admissible (the eigenvalues have a sign) $\Rightarrow$ complicates the proof.
5. Curvature assumption AUBIN 1970 (first proof of C-Y) $\Rightarrow$ hope to withdraw this assumption (used only to estimate uniformly $\lambda(g^{-1}\tilde{g})$).

Reformulation :

$$(E_k) \quad \tilde{\omega}^k \wedge \omega^{m-k} = \frac{e^f}{\binom{m}{k}} \omega^m$$

writes :

$$\underbrace{\ln \sigma_k \lambda}_{\textbf{concave}} \left([\delta_i^j + g^{j\bar{l}} \partial_{i\bar{l}} \varphi]_{1 \leq i,j \leq m} \right) = f \qquad (E_k)$$

It is a nonlinear **elliptic** PDE of second order. It is a complex Hessian equation.

$\rightarrow$ Continuity method : deformation of the equation.

Thanks to Julien KELLER (LATP, Université de Provence), we learned of an independant work : **arXiv Preprint of Zuoliang HOU**, *Complex Hessian equation on Kähler manifolds*, December 2008 ; aiming at the same result as ours, with a different gradient estimate and a similar estimation of $\lambda(g^{-1}\tilde{g})$, **but no proofs given for the C^0 and the C^2 estimates**.

+ **arXiv Preprint of Zuoliang HOU, Xi-Nan MA, Damin WU**, *Complex Hessian equations on compact Kähler manifolds*, December 2008 (without the curvature assumption), **article withdrawn (error found)**.

Theorem (Calabi-Yau)

Si (M, J) complexe ***compacte connexe*** *de $\dim_{\mathbb{C}} m \geq 2$.*
On suppose que M admet des métriques kählériennes compatibles avec la structure complexe fixée au départ.
Si g est une telle métrique kählérienne, ω, $\mathcal{R}_\omega$, alors pour toute 2-forme $\tilde{\mathcal{R}}$ réelle fermée de type (1-1) appartenant à $2\pi\, C_1(M)$ il existe une unique métrique kählérienne $\tilde{g}$ telle que :

- $[\tilde{\omega}] = [\omega]$
- $\mathcal{R}_{\tilde{\omega}} = \tilde{\mathcal{R}}$

Theorem (Equation de Calabi-Yau)

Etant donné $f : M \to \mathbb{R}$ de classe C^∞ telle que $\int_M e^f \omega^m = \int_M \omega^m$,
il existe une unique fonction $\varphi : M \to \mathbb{R}$ de classe C^∞ telle que :

1. $\int_M \varphi \omega^m = 0$;
2. $\tilde{\omega}^m = e^f\, \omega^m \quad (E_m) \quad \left(\frac{\det(\tilde{g})}{\det g} = e^f \right)$.

En outre la solution φ est admissible.

HISTOIRE : Conjecture de Calabi 1954. Aubin (1970, 1976). Preuve Yau 1977.

Th. Aubin, *Métriques riemanniennes et courbures*, J. Diff. Geom. 4 : 383-424 (1970)

Th. Aubin, *Equations du type Monge-Ampère sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris 283 A : 116-120 (1976)

Z. Blocki, *Weak solutions to the complex Hessian equation*, Ann. Inst. Fourier, Grenoble. 55 : 1735-1756 (2005)

L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III : Functions of the eigenvalues of the Hessian* , Acta Math. 155 : 261-301 (1985)

Ph. Delanoë, *Sur l'analogue presque-complexe de l'équation de Calabi-Yau*, Osaka J. Math. 33 : 829-846 (1996)

D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer. Reprint of the 1998 Edition (2001)

Z. Hou, *Complex Hessian equation on Kähler manifold*, Preprint (24 December 2008)

A. Jbilou, *Equations hessiennes complexes sur des variétés kählériennes compactes*, C. R. Acad. Sci. Paris 348 : 41-46 (2010)

S.-Y. Li, *On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian*, Asian J. Math. 8 : 87-106 (2004)

A. Vinacua, *Nonlinear elliptic equations and the complex Hessian*, Comm. PDE 13 : 1467-1497 (1988)

S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations.I.* , Comm. Pure Appl. Math. 31 : 339-441 (1978)