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On the Resolvent Estimates of some Evolution Equations and Applications

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1 Motivation

We consider the problem which consists on finding $u(t)$, a causal distribution of time ($u(t) = 0 \ \ \forall t < 0$) such that

$$\partial_t^2 u + Pu = f \ \ \forall t \in \mathbb{R}, \tag{1}$$

where f takes into account the initial condition and $P = -\Delta$.
Consider the Laplace transform of $u(t)$ formally defined by

$$\hat{u}(\lambda) = \mathcal{L}(u) := \int_0^{+\infty} e^{-i\lambda t} u(t) dt, \quad \text{Im}\lambda < 0$$

Applying the Laplace transform to (1), yields

$$(-\lambda^2 + P)\hat{u}(\lambda) = \hat{f}(\lambda) \iff \hat{u}(\lambda) = R(\lambda^2)\hat{f}(\lambda).$$

The resolvent $R(z) := (P - z)^{-1}$ of P is defined everywhere in complex z-plane except on the spectrum of P (contained in $\mathbb{R}^+$).

The inverse Laplace transform :

$$u(t) = \mathcal{L}^{-1}(\hat{u}) := \frac{1}{2i\pi} \int_{i\lambda_0 + \mathbb{R}} e^{i\lambda t} R(\lambda^2) \hat{f}(\lambda) d\lambda \quad \text{with } \lambda_0 < 0 \tag{2}$$

What happens to the expression (2) when the integration path $i\lambda_0 + \mathbb{R}$ moves from the causal half plane $\operatorname{Im} \lambda < 0$, in which the resolvent is analytic, toward the anti-causal half-plane $\operatorname{Im} \lambda > 0$?

The resonances are the poles of the resolvent extends in the whole complex λ-plane and we have for any fixed compact set $K \subset \Omega$,

$$u(t,x) \sim \sum_{\lambda \in \text{Res}(P)} e^{it\lambda} w_\lambda(x)$$

$$x \in K, \quad t \longrightarrow \infty.$$

2 Wave Equation in $\mathbb{R}^d$

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) = 0 \quad \text{on } \mathbb{R}\times\mathbb{R}^d \\ u(0) = u_0, \\ \partial_t u(0) = u_1 \end{cases}$$

Let $R > 0$, support $(u_0, u_1) \subset B_R = \{x; |x| < R\}$,
Local energy :

$$E_R(u)(t) = \frac{1}{2}\int_{B_R} (|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2)\, dx$$

- d odd :

$$\text{Huygens Principal} \implies E_R(t) = 0 \ \forall t > 2R.$$

- d even :

$$\text{Generalized Huygens Principal} \implies E_R(u)(t) \leq \frac{1}{t^{2d}} E_R(u)(0).$$

3 In exterior domain

- $\mathcal{O}$ obstacle (compact in $\mathbb{R}^d$)
- $\Omega = \mathbb{R}^d \backslash \mathcal{O}$ exterior domain

$$(E) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 & \text{on } \mathbb{R}\times\Omega \\ u(0) = u_0, \partial_t u(0) = u_1 & \text{in } \Omega. \\ u_{/\mathbb{R}\times\partial\Omega} = 0 \end{cases}$$

- Lax-Phillips:

$$E_R(u)(t) \underset{t\to+\infty}{\longrightarrow} 0$$

- Uniform decay of $E_R(t)$?

$$E_R(u)(t) \leq f(t) E_R(u)(0), \quad \text{where } f(t) \underset{t\to+\infty}{\longrightarrow} 0$$

Conjecture of Lax-Phillips (67) : $E_R(t) \searrow 0$ uniformly $\iff$ Ω is non trapping.

- $'' \implies ''$Ralston (1969).
- $'' \impliedby ''$
  - Morawetz (61), Strauss (75),... (tech. multip.) : Start shape, strictly convex...
  - Wilcox (75), (59), Vainberg (75), Rauch (78) : Resolvent.
  - Melrose (79): Theorem of propagation of singularity of Melrose- Sjöstrand
- Vodev (99):

$$\text{if } \exists \ f \text{ such that } E_R(t) \leq f(t)E(0), \ \text{ where } f(t) \underset{t \to +\infty}{\longrightarrow} 0$$

then

$$\exists \lambda_0 > 0, c > 0 \text{ such that } \forall |\lambda| \geq \lambda_0, \|\lambda R(\lambda)\|_{L^2_{comp} \to L^2_{loc}} \leq c,$$

$$\begin{cases} E_R(t) \leq ce^{-\alpha t}E(0) & \text{if } d \text{ odd} \\ E_R(t) \leq \frac{c}{t^{2d}}E(0) & \text{if } d \text{ even} \end{cases}$$

3.1 Internal stabilization for wave equation

Let $(u_{0,} u_1) \in H = H_D \times L^2$

$$(A) \qquad \begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u = 0 & \text{on } \mathbb{R}_+ \times \Omega \\ u(0) = u_0, \partial_t u(0) = u_1 & \text{in } \Omega. \\ u_{/\mathbb{R}_+ \times \partial\Omega} = 0 \end{cases}$$

$a(x) \in C_0^\infty(\Omega, \mathbb{R}_+)$. We denote $\omega = \{x \in \Omega / a(x) > 0\}$.
The total energy $E(t)$ of the solution :

$$E(t) = \frac{1}{2} \int_\Omega (| \nabla u(t,x) |^2 + | \partial_t u(t,x) |^2) \, dx. \tag{3}$$

$$E(t_2) - E(t_1) = - \int_{t_1}^{t_2} \int_\Omega a(x) \, |\partial_t u(t,x)|^2 \, dx dt \leq 0, \quad 0 \leq t_1 < t_2. \tag{4}$$

- Zuazua (91): Stabilization of the global energy (dissipater localized near $\partial\Omega$ and near the infinity.
- Bardos, Lebeau, Rauch (92): bounded domain
  (Control geometric Condition : "Every generalized geodesic meet ω").

Definition 1 (Exterior Geometric Control (E.G.C.)) *Let $R > 0$ be such that $O \subset B_R$. We say that ω verifies the Exterior Geometric Control condition on B_R (E.G.C.) if there exists $T_R > 0$ such that every generalized bicharacteristic γ starting from B_R at time $t = 0$, is such that:*
$*)$ γ leaves $\mathbb{R}^+ \times B_R$ before time T_R, or
$*)\gamma$ meets $\mathbb{R}_+ \times \omega$ between the time 0 and T_R.

3.1.1 In odd dimension

Theorem 1 *Let $R > 0$ such that ω verifies the E.G.C. on B_R, then there exists $c > 0$, $\alpha > 0$ such that*

$$E_R(u(t)) \leq ce^{-\alpha t}E(0) \tag{5}$$

for all solution u of (A) with $(u_0, u_1) \in H$ supported in B_R.

0 Aloui (L.)& Khenissi (M.), *St*abilisation de l'équation des ondes dans un domaine extérieur, Rev.Math. Iberoamericana, 28 (2002), 1-16.

- Lax - Phillips theory adapted to the case of damped wave equation

$$\|Z(t)\| \leq e^{-\beta t} \iff E_R(t) \leq e^{-\alpha t} E_R(0)$$

- Microlocal Analysis: Theorem of propagation of microlocal defect measure (Gérard, Lebeau, Burq)).

3.1.2 In even dimension

In order to generalize the last result to even dimension, Khenissi [1] has studied the outgoing resolvent $(\lambda^2 + \Delta + i\lambda a(x))^{-1}$. He proved that this resolvent is bounded as an operator from $L^2_{comp}(\Omega)$ to $H^1_{loc}(\Omega)$ on a strip of the shape { $\mathrm{Im}\,\lambda < c_1,\ |\lambda| > c_2$; $c_1, c_2 > 0$}

[1] Khenissi (M.)- *Equation des ondes amorties dans un domaine extérieur,* Bull.Soc. Math. France, 131 (2) (2003), 211-228.

Theorem 1 *For all $\delta < C(\infty)$, there exists $\lambda_0 > 0$ so that the cutoff resolvent $R_\chi(\lambda)$ is analytic in the region*

$$\{\lambda \in \mathbb{C},\ \Im m\lambda \leq \delta,\ |\Re e\lambda| \geq \lambda_0\}.$$

and satisfies there the estimate

$$\|\nabla R(\lambda) f\|^2_{L^2_R} + \|\lambda R(\lambda) f\|^2_{L^2_R} \leq c\, \|f\|^2_{L^2}\,. \tag{6}$$

for all $f \in L^2$, Supp $f \subset B_R$

- If Ω is non trapping then $C(\infty) = +\infty$.
- If Ω is trapping then $C(\infty) \leq ||a||_\infty$
  - without E.G.C then $C(\infty) = 0$.
  - with E.G.C then $C(\infty) > 0$.

E.G.C.$\Rightarrow$ uniform decay of the local energy.

$$\begin{aligned} E_R(t) &\leq ce^{-\alpha t} E(0),\ \forall\, t > 1 \ \text{ in odd space dimension,} \\ E_R(t) &\leq \frac{c}{t^{2d}} E(0),\ \forall\, t > 1 \ \text{ in even space dimension.} \end{aligned} \tag{7}$$

3.2 Boundary Stabilization

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in} \quad]0,+\infty[\times\Omega, \\ \partial_\nu u + \underline{a(x)\partial_t u} = 0 & \text{on} \quad]0,+\infty[\times\partial\Omega, \\ u(0,x) = u_0, \partial_t u(0,x) = u_1 & \text{in} \quad \Omega, \end{cases} \tag{8}$$

$a(x) \in C^\infty(\partial\Omega)$ is a non negative real-valued function.

$$u \in C([0,+\infty[, H_D) \cap C^1([0,+\infty[, L^2(\Omega)).$$

The energy identity

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_{\partial\Omega} a(x)\left|\partial_t u(t,x)\right|^2 dxdt \leq 0, \quad 0 \leq t_1 < t_2. \tag{9}$$

3.2.1 Distributions of the resonances near the real axis

Let $\chi \in C_0^\infty(R^n), \chi = 1$ on O and denote

$$\tilde{\mathbb{C}} := \begin{cases} \mathbb{C} & d \geq 3 \text{ odd} \\ \{\lambda \in \mathbb{C}\,;\ -\frac{3\pi}{2} < \arg\lambda < \frac{\pi}{2}\} & d \geq 4 \text{ even.} \end{cases}$$

The cutoff resolvent

$$R_\chi(\lambda) := \chi R(\lambda)\chi$$

considered as operator from $L^2(\Omega)$ to $L^2(\Omega)$, holomorphic on $\{\operatorname{Im}\lambda < 0\}$ extends meromorphically to $\tilde{\mathbb{C}}$ with poles in $\{\operatorname{Im}\lambda > 0\}$ (resonances).

Theorem 2 *Suppose that* $\Gamma_0 = \{x \in \partial\Omega, a(x) > 0\}$ *satisfies the E.G.C., then there exist positive constants* α, β *such that* $R_\chi(\lambda)$ *has no poles in the region*

$$\Lambda_{\alpha,\beta} = \{\lambda \in \mathbb{C} \ / \ \ \mathrm{Im}\,\lambda \leq \alpha; |\mathrm{Re}\,\lambda| \geq \beta\}.$$

Furthermore, there exists $c > 0$ *such that for any* $f \in L^2_R(\Omega) := \{g \in L^2(\Omega)/Supp(g) \subset B(0,R)\}$ *and* $\lambda \in \Lambda_{\alpha,\beta}$

$$\|\nabla R(\lambda) f\|_{L^2(\Omega_R)} + \|\lambda R(\lambda) f\|_{L^2(\Omega_R)} \leq c\,\|f\|_{L^2(\Omega)}. \tag{10}$$

Corollary 1 *Under the hypotheses of Theorem 1, there exist* $\alpha, c > 0$ *such that for all solution of (8) with initial data in* H_r, *we have*

$$\begin{aligned} E_R(t) &\leq ce^{-\alpha t} E(0),\ \forall\, t > 1 \ \text{ in odd space dimension,} \\ E_R(t) &\leq \frac{c}{t^{2d}} E(0),\ \forall\, t > 1 \ \text{ in even space dimension.} \end{aligned} \tag{11}$$

[1] L. Aloui end M. Khenissi, Boundary stabilization of the wave and Schrödinger equations in exterior domains, DCDS-A, Volume 27, Number 3, July 2010

- So the high-frequency behavior (i.e. the large pole-free regions near the real axis) decides whether there is uniform local energy decay or not.
- The explicit rate is calculated from the low-frequency asymptotic.
- We show that it has a meromorphic extension to $\tilde{\mathbb{C}}$. Then we prove, for low frequencies, that $R(\lambda)$ has the same behavior as that of the free resolvent $R_0(\lambda)$ near $\lambda = 0$.
- We prove Theorem 1. We argue by contradiction we obtain two sequences $\lambda_n \to \infty$ and (u_n) satisfying

$$\|\nabla u_n\|^2_{L^2(\Omega_R)} + \|\lambda_n u_n\|^2_{L^2(\Omega_R)} = 1, \|\lambda_n a u_n\|_{L^2(\Gamma)} \to 0. \quad (12)$$

  Multiplying u_n by $e^{-i\lambda_n t}$, we obtain a sequence (v_n) solutions of a damped wave equation such that

$$v_n \to 0 \text{ in } H^1_{loc}(\mathbb{R}_+ \times \Gamma_0) \text{ and } \partial_\nu v_n \to 0 \text{ in } L^2_{loc}(\mathbb{R}_+ \times \Gamma_0). \quad (13)$$

  Using the Lifting Lemma and the propagation of the microlocal defect measures in the positive sense of time we can propagate the strong convergence (13) to the region $]T_R, +\infty[\times\Omega_R$. Translating this result to (u_n) we get a contradiction.

4 Stabilization of Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0 & \text{in }]0,+\infty[\times\Omega, \\ \partial_\nu u + ia(x)u = 0 & \text{on }]0,+\infty[\times\partial\Omega, \\ u(0,.) = f & \text{in } \Omega, \end{cases} \tag{14}$$

where a is as above and $f \in L^2(\Omega)$.

- $\Omega = \mathbb{R}^d$: Local energy

$$E_R(u)(t) := \|u(t,.)\|_{L^2(B_R)} \leq \frac{c}{t^{d/2}} \|f\|_{L^2}, \quad t > 0$$

- Ω non-trapping domain
  - Vainberg (73): $E_R(t) \searrow \frac{1}{t}$ if d is even and $\searrow \frac{1}{t^{3/2}}$ if d is odd.
  - K. Tsutsumi (84): $E_R(t) \searrow \frac{1}{t^{d/2}}$:

$$(-\lambda^2 - \Delta)^{-1} \implies (-i\tau - \Delta)^{-1}$$

- In the trapping case
  - Ralston gives an example with a sequence of poles of $(\lambda^2 + \Delta)^{-1}$ converging exponentially to the real axis.

In order to solve this situation and to make the decay of the local energy uniform, the authors Al-Kh[2] have been interested to the dissipative Schrödinger equation with an internal damping term. Under the E.G.C, they proved an estimate on the cut-off resolvent $\chi(-i\tau - \Delta + ia)^{-1}\chi$ and they thus deduced the uniform decay of the local energy.

- $\tilde{R}(\tau)$ the outgoing resolvent associated to the problem (14)

$$\tilde{R}(\tau)f = i\int_0^{+\infty} e^{\tau t}u(t)dt, \quad \mathrm{Re}\,\tau < 0, \tag{15}$$

with $u(t)$ solution of (14). It is clear that the relation (15) defines a family of bounded operators from $L^2(\Omega)$ to $L^2(\Omega)$, holomorphic on $\{\mathrm{Re}\,\tau < 0\}$.

[2] Aloui (L.) and Khenissi (M.)- *Stabilization of Schrödinger equation in exterior domains*, Control, Optimisation and Calculus of Variations, ESAIM : COCV. 13 No 3 (2007), 570-579.

Theorem 3 *Suppose that Γ_0 satisfies the E.G.C., then there exist positive constants α, β such that $\tilde{R}_\chi(\tau)$ has no pole in the region*

$$\Gamma_{\alpha,\beta} = \{\tau \in \mathbb{C} \;\; / \;\; \mathrm{Re}\,\tau \le \alpha; |\mathrm{Im}\,\tau| \ge \beta\}\,.$$

Furthermore, for two positive constants r and r' with $r', r > R$ there exists $c > 0$ such that for any $g \in L^2_r(\Omega)$ and $\tau \in \Gamma_{\alpha,\beta}$

$$\left\|\tilde{R}_\chi(\tau) g\right\|_{L^2(\Omega_{r'})} \le c\, \|g\|_{L^2(\Omega)}\,. \tag{16}$$

Corollary 2 *Under the hypotheses of the Theorem 2, for two positive constants $r', r > R$ there exists $c > 0$ such that for all solution u of (14) with initial data f in $L^2_r(\Omega)$ we have*

$$\|u(t,.)\|_{L^2(\Omega_{r'})} \le \frac{c}{t^{d/2}}\, \|f\|_{L^2(\Omega)}\,, \forall\, t > 1.$$

We present a simple method allowing us to see the resolvent of the Schrödinger equation as a perturbation of that of the wave equation.

[2] L. Aloui end M. Khenissi, Boundary stabilization of the wave and Schrödinger equations in exterior domains, DCDS-A, Volume 27, Number 3, July 2010

5 Smoothing effect

It is well known that the Schrödinger equation enjoys some smoothing properties. One of them says that if $u_0 \in L^2(\mathbb{R}^d)$ with compact support, then the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^d \\ u(0,.) = u_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{17}$$

satisfies

$$u \in C^\infty(\mathbb{R} \setminus \{0\} \times \mathbb{R}^d).$$

We say that the Schrödinger propagator has an infinite speed. Another type of gain of regularity for system (17) is the Kato-$1/2$ smoothing effect, namely any solution of (17) satisfies

$$\int_{\mathbb{R}} \int_{|x|<R} |(1-\Delta)^{\frac{1}{4}} u(t,x)|^2 dx dt \leq C_R \|u_0\|^2_{L^2(\mathbb{R}^d)} . \tag{18}$$

In particular, this result implies that for a.e. $t \in \mathbb{R}$, $u(t,.)$ is locally smoother than u_0 and this happens despite the fact that (17) conserves the global L^2 norm.

The Kato-effect has been extended to

- Variable coefficients operators with non trapping metric by Doi(96)
- Non trapping exterior domains by Burq, Gerard and Tzvetkov (04).
- On the other hand, Burq (04) proved that the nontrapping assumption is necessary for the $H^{1/2}$ smoothing effect.

Recently, we have introduced the forced smoothing effect for Schrödinger equation. The idea is inspired from the stabilization problem and it consists of acting on the equation in order to produce some smoothing effects. More precisely, the following regularized Schrödinger equation on a bounded domain $\Omega \subset \mathbb{R}^d$ is considered:

$$\begin{cases} i\partial_t u - \Delta u + ia(x)(-\Delta)^{\frac{1}{2}}a(x)u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u(0,.) = f & \text{in } \Omega, \\ u|_{\mathbb{R}\times\partial\Omega} = 0, \end{cases} \tag{19}$$

Under the geometric control condition (G.C.C.) on the set $w = \{a \neq 0\}$, it is proved by Aloui[3], that any solution with initial data in $L^2(\Omega)$ belongs to $L^2_{loc}((0,\infty), H^1(\Omega))$. Then by iteration of the last result, a C^∞-smoothing effect is proved.

[3] ALOUI (L.)- Smoothing effect for regularized Schrödinger equation on bounded domains, Asymptotic Analysis 59- 2008

Note that these smoothing effects hold away from $t = 0$ and they seem strong compared with the Kato effect for which the GCC is necessary. Therefore the case when $w = \{a \neq 0\}$ does not control geometrically Ω is very interesting.

In this work we give an example of geometry where the geometric control condition is not satisfied but the C^∞ smoothing effect holds.
Let $O = \cup_{i=1}^{N} O_i \subset \mathbb{R}^d$ be the union of a finite number of bounded strictly convex bodies, O_i, satisfying the conditions of Ikawa.
Let B be a bounded domain containing O with smooth boundary such that $\Omega_0 = O^c \cap B$ is connected, where $O^c = \mathbb{R}^d \setminus O$.
In the present work, we will consider the regularized Schrödinger equation (19) in Ω_0.

Our main result is the following

Theorem 2 *Assume $a \in C^\infty(\Omega_0)$ is constant near the boundary of B. Then there exist positive constants σ_0 and c such that for any $|\operatorname{Im}\tau| < \sigma_0$ and $f \in L^2(\Omega_0)$*

$$\left\|(-\Delta_D - \tau + ia(x)(-\Delta_D)^{\frac{1}{2}}a(x))^{-1}f\right\|_{L^2(\Omega_0)} \leq C\frac{\log^2\langle\tau\rangle}{\langle\tau\rangle^{\frac{1}{2}}}\|f\|_{L^2(\Omega_0)}, \qquad (20)$$

where $\langle\tau\rangle = \sqrt{1+|\tau|^2}$.

Note also that a better bound (with $\log$ instead of $\log^2$) was obtained by Christainson (07-10) in the case of the damped wave equation on compact manifolds without boundary under the assumption that there is only one closed hyperbolic orbit which does not pass through the support of the dissipative term. This has been recently improved by Schenck(09) for a class of compact manifolds with negative curvature, where a strip free of eigenvalues has been obtained under a pressure condition.

[3] L. Aloui, G. Vodev and M. Khenissi, Smoothing effect for the regularized Schrödinger equation with non controlled orbits, submited to publication in J.Diff. Eq.

As an application of this resolvent estimate we obtain the following smoothing result for the associated Schrödinger propagator.

Theorem 3 *Let $s \in \mathbb{R}$. Under the hypothesis of Theorem 2, we have*

(i) For each $\varepsilon > 0$ there is a constant $C > 0$ such that u, defined by $u(t) = \int_0^t e^{i(t-\tau)A_a} f(\tau) d\tau$ satisfies

$$\|u\|_{L^2_T H^{s+1-\varepsilon}(\Omega_0)} \leq C \|f\|_{L^2_T H^s(\Omega_0)} \tag{21}$$

for all $T > 0$ and $f \in L^2_T H^s(\Omega_0)$.

(ii) If $v_0 \in H^s(\Omega_0)$ then

$$v \in C^\infty((0, +\infty) \times \Omega_0) \tag{22}$$

where v is the solution of (19) with initial data v_0.

Theorem 2 also implies the following stabilization result.

Theorem 4 *Under the hypotheses of Theorem 2, there exist* $\alpha, c > 0$ *such that for all solution* u *of (19) with initial data* u_0 *in* $L^2(\Omega_0)$, *we have*

$$\|u\|_{L^2(\Omega_0)} \leq ce^{-\alpha t}\|u_0\|_{L^2(\Omega_0)},\ \forall\ t > 1.$$

This result shows that we can stabilize the Schrödinger equation by a (strongly) dissipative term that does not satisfy the geometric control condition of B. L. R. In fact, to have the exponential decay above it suffices to have the estimate (20) with a constant in the right-hand side.

Thank you !!