Existence of hypersurfaces with large constant mean curvature and free boundaries

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Introduction

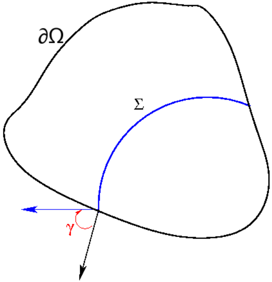
Let Ω be an open smooth subset of $\mathbb{R}^{m+1}$, $m \geq 2$.
We are interested in the existence of embedded constant mean curvature hypersurfaces Σ into Ω with non empty boundary such that

$$\partial\Sigma \subset \partial\Omega \tag{1}$$

and which

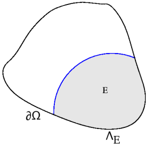
$$\text{intersect } \partial\Omega \text{ at a constant angle } \gamma \in (0, \pi). \tag{2}$$

Such hypersurfaces are called Capillary hypersurfaces in Ω.



Physical motivations

Capillary surfaces correspond to the physical problem of the behavior of an incompressible liquid in a container Ω in the absence of gravity.



They are critical points of an energy functional under two constraints:

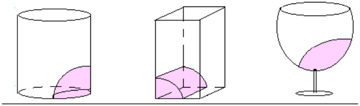
$$\begin{array}{ll} \text{Crit} & (\mathcal{P}(E,\Omega) - \cos(\gamma)\,\mathrm{Area}(\Lambda_E)) \\ E \in \mathfrak{C} & \\ |E| = v, & \end{array}$$

where $\mathfrak{C}$ is the class of sets $E \subset \Omega$ such that ∂E divides Ω in two connected components with $\partial\partial E \subset \partial\Omega$ and $\Lambda_E \subset \partial\Omega$ the boundary of one of these components.

Isoperimetric Problem

Let Ω be an open subset of $\mathbb{R}^{m+1}$, the Isoperimetric Problem is the minimum problem:

$$\min_{\substack{E \subset \Omega \\ |E| = v}} \mathcal{P}(E, \Omega)$$



Variational Consideration

Let $\{\Psi_t\}_t$ be a one parameter family of diffeomorphisms defined on $\mathbb{R}^{m+1}$. Denote

$$\zeta = \frac{\partial \Psi_t}{\partial t}_{|t=0}.$$

If $E \in \mathfrak{C}$, let $\Sigma := \partial E \cap \Omega$.

A variation is called admissible if

$$\Psi_t(\operatorname{int}\Sigma) \subset \operatorname{int}\Omega \qquad \text{and} \qquad \Psi_t(\partial\Sigma) \subset \partial\Omega \quad \text{for any } t$$

and volume-preserving if

$$|\Psi_t(E)| = |E| \qquad \text{for every } t.$$

An admissible variation induces $E_t = \Psi_t(E) \subset \Omega$ and $\Lambda_{E_t} \subset \partial\Omega$.
We consider the total energy:

$$\mathcal{E}(t) := \mathcal{P}(E_t, \Omega) - \cos(\gamma)\,\mathrm{Area}(\Lambda_{E_t}). \tag{3}$$

Definition

We say that a set $E \in \mathfrak{C}$ is critical (or stationary) for the total energy:

$$\mathcal{E} = \mathcal{E}(0) = \mathcal{P}(E, \Omega) - \cos(\gamma)\, Area(\Lambda_E). \tag{4}$$

If $\mathcal{E}'(0) = 0$ for any admissible volume-preserving variation.

If E is critical, we call the hypersurface $\Sigma := \partial E \cap \Omega$ a Capillary hypersurface.

First variation

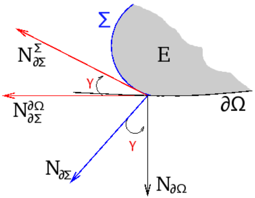
The first variation of area and volume yields ($\Sigma = \partial E \cap \Omega$):

$$\left.\frac{d\,\mathcal{P}(E_t,\Omega)}{dt}\right|_{t=0} = -\int_\Sigma m H_\Sigma \langle \zeta, N_\Sigma\rangle dA + \oint_{\partial\Sigma} \langle \zeta, N_{\partial\Sigma}^{\Sigma}\rangle ds; \quad (5)$$

$$\left.\frac{d\,\mathrm{Area}(\Lambda_{E_t})}{dt}\right|_{t=0} = \oint_{\partial\Sigma} \langle \zeta, N_{\partial\Sigma}^{\partial\Omega}\rangle ds; \quad (6)$$

where

- H_Σ is the mean curvature of Σ,
- N_Σ is the unit outer normal vector-field along Σ,
- $N_{\partial\Sigma}^{\Sigma}$(resp. $N_{\partial\Sigma}^{\partial\Omega}$) is the unit normal vector-field along $\partial\Sigma$ in Σ (resp. $\partial\Omega$) (see figure...).



Consequences

From the above, we deduce that

$$\mathcal{E}'(0) = -\int_{\Sigma} n H_{\Sigma} \langle \zeta, N_{\Sigma} \rangle dA + \oint_{\partial\Sigma} \langle \zeta, N_{\partial\Sigma}^{\Sigma} \rangle ds - \cos(\gamma) \oint_{\partial\Sigma} \langle \zeta, N_{\partial\Sigma}^{\partial\Omega} \rangle ds = 0 \tag{7}$$

while

$$\left.\frac{d\,|E_t|}{dt}\right|_{t=0} = \int_{\Sigma} \langle \zeta, N_{\Sigma} \rangle dA = 0. \tag{8}$$

- Choosing interior normal variations: $\zeta = \omega N_{\Sigma}$ together with volume preserving ($\int_{\Sigma} \omega \, dA = 0$) implies that

$$H_{\Sigma} \equiv Const. \qquad \text{in } \Sigma.$$

- Choosing boundary normal variations: $\zeta = \omega N_{\partial\Sigma}^{\partial\Omega}$ implies that

$$\langle N_{\partial\Sigma}^{\partial\Omega}, N_{\partial\Sigma}^{\Sigma} - \cos(\gamma)\, N_{\partial\Sigma}^{\partial\Omega} \rangle = 0 \qquad \text{on } \partial\Sigma \subset \partial\Omega. \tag{9}$$

At the equilibrium

A hypersurface $\Sigma \subset \Omega$ is Capillary if it has constant mean curvature and intersect $\partial\Omega$ with an angle γ along its boundary in the sense that

- $\langle N^{\Sigma}_{\partial\Sigma}, N^{\Omega}_{\partial\Sigma} \rangle = \cos(\gamma)$ or equivalently
- $\langle N_{\Sigma}, N_{\Omega} \rangle = \cos(\gamma)$, where N_{Ω} is the normal of $\partial\Omega$.

- Conclusion

  We conclude that the Euler-Lagrange equations reads:

$$\begin{cases} H_{\Sigma} & = & Const. & \Sigma \\ \partial\Sigma & \subset & \partial\Omega & \\ \langle N^{\Sigma}_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma} \rangle & = & \cos(\gamma) & \partial\Sigma. \end{cases} \tag{10}$$

Even thought the direct method of the calculus of variation gives existence of minimizers, the complete description of geometry, topology of these surfaces is far from being complete. One can see for instance
Ros-Vergasta or Ros-Souam where they give the geometric structures of stable Capillary hypersurfaces

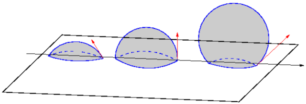
- Ros A. and Souam R., On stability of capillary surfaces in a ball, Pacific J. Math. 178 (1997) 345- 361.
- Ros A. and Vergasta E., Satability for hypersurfaces of constant mean curvature with free boundary, Geom. Dedicata 56 (1995), no. 1, 19-33.

Some Examples

- For any angle $\gamma \in (0, \pi)$, there is a Capillary spherical cap S^n_γ with mean curvature $H = 1$ in $\mathbb{R}^{n+1}_+ + \cos\gamma E_{n+1}$. We can parameterize it by the inverse of the stereographic projection $\Theta : \mathbb{R}^n \to S^n$ by

$$\Theta(z) = \left(\frac{2\,z^1}{1+|z|^2}, \ldots, \frac{2\,z^n}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right).$$

The restriction of $\Theta\Big|_{B(0, \frac{1-\cos\gamma}{1+\cos\gamma})}$ parameterize the spherical cap S^n_γ.



- If $\Omega = \mathbb{R}^m_+$ with $1 \le k < m$ then the cylindrical cap $rS^n_\gamma \times \mathbb{R}^k$ around $\mathbb{R}^k$ is a Capillary hypersurface, where $n := m - k$ with constant mean curvature $H = \frac{n}{rm}$.

The Problem as a Geometric one

We can reformulate the question of finding critical point of $\mathcal{E}$ to a **prescribed mean curvature free boundary problem**:
for a given real number H and an angle γ, find a hypersurface Σ (with prescribed topology) satisfying the following conditions:

$$(GMP)\begin{cases} H_\Sigma & \equiv & H \quad \text{in } \Sigma, \\ \partial\Sigma & \subset & \partial\Omega, \\ \langle N^{\Sigma}_{\partial\Sigma}, N^{\Omega}_{\partial\Sigma}\rangle & = & \cos\gamma \quad \text{on } \partial\Sigma, \end{cases}$$

A more general one is to prescribe a non-constant mean curvature function $H(p)$ and angle $\gamma(\sigma)$.

One related PDE problem

A particular case when prescribing the topology of a disc is the Free Boundary Plateau Problem for H-surfaces.

- Suppose Σ is parameterized by a map $u \in C^2(B; \mathbb{R}^3) \cap C^1(\bar{B}; \mathbb{R}^3)$ over the unit disc

  $$B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

  The above (GMP) then is equivalent to the problem:

- $$\begin{cases} \Delta u = 2H u_x \wedge u_y & \text{in } B, \qquad \text{(mc equation)} \\ |u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y & \text{in } B, \qquad \text{(conformality)} \end{cases} \tag{11}$$

- $$\begin{cases} u(\partial B) \subset \partial\Omega, & \\ \frac{\partial u}{\partial n}(\sigma) \perp T_{u(\sigma)}\partial\Omega & \forall \sigma \in \partial B. \end{cases} \qquad \text{(free boundary)} \tag{12}$$

Invariance with the conformal group of the disc

One of the main features in the study of the variational problem associated to the (FBPP), is the lack of compactness due to the invariance under the action of the non-compact group of conformal transformations of the unit disc: The Möbius Group

$$G = \left\{ g_{\theta,a}(X) = e^{i\theta} \frac{X - a}{1 - \bar{a}X}, \quad \theta \in [-\pi, \pi), \quad a = (a_1, a_2) \in B \right\}. \tag{13}$$

One needs new tools for the study of this problem.

Results obtained on the study of (FBPP)

- M.Struwe [13] proved existence of minimal (H=0) solutions (not necessarily embedded).
- M.Struwe [14] proved existence of solutions (not necessarily embedded) for almost every H bounded.
- W.Bürger and E. Kuwert [3] proved that inf-minimums are always achieved and they are union of finitely many discs.
- We have obtained a result somehow complementary to Struwe's own: proving that there exits a family of solutions concentrating at a non-degenerate minimal submanifold of $\partial\Omega$ as $H \to \infty$. (When $K = Q$ a point of $\partial\Omega$, $\exists\, u_H$ converging to $Q \in \partial\Omega$ provided Q is a stable critical point of the mean curvature of $\partial\Omega$).

When $K = Q$ a point of $\partial\Omega$ the result can be proved by adopting a variational perturbation method, see [5]. The technique goes back to Ambrosetti-Badiale [1] and successfully used by many authors in a nearly context. Notably one can see the works of Caldiroli-Musina [4] and Felli [7] in the following perturbed H-Bubble problem

$$\begin{cases} \Delta u = 2(H_0 + \varepsilon H_1(u))\, u_x \wedge u_y & \text{in } \mathbb{R}^2, \\ \int_B |\nabla u|^2 < \infty. \end{cases}$$

Second Variation

Let Σ be capillary hypersurface and denote by B_Σ its second fundamental form. The Jacobi operator (or the linearized mean curvature operator about Σ) is given by the second variation of the total energy functional $\mathcal{E}$.
For any volume-preserving admissible variation, we have

$$\mathcal{E}''(0) = -\int_\Sigma \left(\omega\Delta_\Sigma\omega + |B_\Sigma|^2\omega^2\right) dA + \oint_{\partial\Sigma} \left(\omega\frac{\partial\omega}{\partial\eta} - q\,\omega^2\right) ds, \quad (14)$$

where $\eta = N_{\partial\Sigma}^{\Sigma}$ and $\omega = \langle\zeta,\, N_\Sigma\rangle$ and

$$q = \frac{1}{\sin(\gamma)}\, B_{\partial\Omega}(N_{\partial\Sigma}^{\partial\Omega}, N_{\partial\Sigma}^{\partial\Omega}) - \cot(\gamma)\, B_{\partial\Sigma}(\eta,\eta). \quad (15)$$

The Jacobi Operator

By Barbosa-Do Carmo [2], for any smooth ω with $\int_\Sigma \omega dA = 0$ there exits an admissible, volume-preserving variation with variation vector field $\omega\, N$ as a normal part.
The Jacobi operator about Σ is defined for any $\omega, \omega' \in H^1(\Sigma)$ by

$$\langle \mathfrak{L}_\gamma\, \omega, \omega' \rangle := \int_\Sigma \left\{ \nabla\omega\nabla\omega' - |B_\Sigma|^2 \omega\, \omega' \right\} dA - \oint_{\partial\Sigma} q\, \omega\, \omega' ds, \quad (16)$$

where

$$q = \frac{1}{\sin(\gamma)}\, B_{\partial\Omega}(N^{\partial\Omega}_{\partial\Sigma}, N^{\partial\Omega}_{\partial\Sigma}) - \cot(\gamma)\, B_{\partial\Sigma}(\eta, \eta). \quad (17)$$

The hemisphere S^n_+ ($\gamma = \frac{\pi}{2}$)

In $\mathbb{R}^{n+1}_+ = \Omega$, the Jacobi operator of the Capillary spherical cap is

$$\langle \mathbb{L}_{S^n_+}\,\omega, \omega' \rangle = -\int_{S^n_+} \Big(\Delta_{S^n_+}\omega + n\omega \Big)\,\omega'\,d\sigma + \oint_{S^n_+} \frac{\partial \omega}{\partial \eta}\omega'\,ds. \qquad (18)$$

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Let $\Theta = (\Theta^1, \cdots, \Theta^n, \Theta^{n+1}) : B^n \to S^n_+$ be a parametrization of S^n_+.

By spectral decomposition of $\Delta_{S^n_+}$ (with zero Neumann) we have that

$$\mathrm{Ker}\, \mathbb{L}_{S^n_+} = \mathrm{span}\, \{\Theta^1; \cdots ; \Theta^n\}.$$

The Cylinder $\mathcal{C}_r := rS^n_+ \times \mathbb{R}^k$

In $\mathbb{R}^{m+1}_+ = \Omega$, with $\partial\Omega = \mathbb{R}^m \times \{0\} = \mathbb{R}^n \times \mathbb{R}^k \times \{0\}$.

The Jacobi operator of the Capillary cylindrical cup $\mathcal{C}_r := rS^n_+ \times \mathbb{R}^k$ around $K := \mathbb{R}^k$:

$$r^{2-n}\langle \mathbb{L}_{\mathcal{C}_r}\,\omega, \omega'\rangle = -r^{2-n}\int_{S^n_+\times K}\left(r^2\Delta_K\omega + \Delta_{S^n_+}\omega + n\omega\right)\omega'\,d\sigma dy + \oint_{S^n_+\times K}\frac{\partial\omega}{\partial\eta}\omega'\,ds. \tag{19}$$

Concentration at points

- Letting $p \in \partial\Omega$, consider

$$\bar{\Sigma}_{p,r} := \{q \in \bar{\Omega} : \quad d(q,p) = r\}. \tag{20}$$

- Our goal is to perturb $\bar{\Sigma}_{p,r}$ to a set satisfying our Geometric problem.
- Notice, hopefully, that $\bar{\Sigma}_{p,r}$ satisfies almost the E-L for the Capillary problem with

$$\begin{aligned} r\, H_{\bar{\Sigma}_{p,r}} &= n + \mathcal{O}(r) && \text{in } \bar{\Sigma}_{p,r}, \\ \partial\bar{\Sigma}_{p,r} &\subset \partial\Omega, && \\ \langle N^{\bar{\Sigma}}_{\partial\bar{\Sigma}_{p,r}}, N^{\partial\Omega}_{\partial\bar{\Sigma}_{p,r}} \rangle &= 0 && \text{on } \partial\bar{\Sigma}_{p,r}. \end{aligned} \tag{21}$$

Perturbed hemisphere

As before let $\Theta = (\Theta^1, \dots, \Theta^n, \Theta^{n+1}) : B_n \to S^n_+$ parametrizing S^n_+ and $\tilde{\Theta} := (\Theta^1, \dots, \Theta^n, 0)$.

All surfaces nearby $\bar{\Sigma}_{p,r}$ can be parameterized by a parametric function $\omega : S^n_+ \to \mathbb{R}$:

$$\Sigma_{p,r,\omega} := \exp_p^{\partial\Omega}(r(1+\omega)\tilde{\Theta}) - r(1+\omega)\Theta^{n+1} N_{\partial\Omega}(\cdot) \tag{22}$$

in particular

$$\partial\Sigma_{p,r,\omega} \subset \partial\Omega \qquad \text{because} \qquad \Theta^{n+1} = 0 \quad \text{on } \partial S^n_+ \tag{23}$$

and the initial hypersurface is

$$\bar{\Sigma}_{p,r} = \Sigma_{p,r,0}. \tag{24}$$

Expansions of the mean curvature

The expansions of the mean curvature $H(p, r, \omega)$ of the hypersurface $\Sigma_{p,r,\omega}$ in terms of r and ω yields

-
$$
\begin{aligned}
rH(p, r, \omega) &= n + r\,U(\Theta) + \mathcal{O}(r^2) \\
&- \left(\Delta_{S^n_+}\omega + n\omega\right) + r\,L(\omega) + Q(\omega) \ \text{ in } \ \Sigma_{p,r,\omega};
\end{aligned}
\tag{25}
$$

  where $U(\Theta) \perp \mathrm{Ker}\mathbb{L}_{S^n_+}$;
- The orthogonality condition is equivalent to

$$
\langle N^{\Sigma}_{\partial\Sigma_{p,r,\omega}}, N^{\partial\Omega}_{\partial\Sigma_{p,r,\omega}} \rangle = -\frac{\partial\omega}{\partial\eta} + r^2\,L(\omega) + Q(\omega) \ \text{ on } \ \partial\Sigma_{p,r,\omega}.
\tag{26}
$$

Adjusting the geodesic half-sphere $\bar{\Sigma}_{p,r}$

- Find $\bar{\omega}^p$ such that

$$\begin{aligned} r\,H_{\Sigma_{p,r,r\bar{\omega}^p}} &= n + \mathcal{O}(r^2) \quad \text{in } \Sigma_{p,r,r\bar{\omega}^p}, \\ \partial\Sigma_{p,r,r\bar{\omega}^p} &\subset \partial\Omega, \\ \langle N^{\Sigma}_{\partial\Sigma_{p,r,r\bar{\omega}^p}}, N^{\partial\Omega}_{\partial\Sigma_{p,r,r\bar{\omega}^p}} \rangle &= \mathcal{O}(r^2) \quad \text{on } \partial\Sigma_{p,r,r\bar{\omega}^p}. \end{aligned} \tag{27}$$

- This is equivalent to solve

$$\mathbb{L}_{S^n_+}[\bar{\omega}^p] = U(\Theta). \tag{28}$$

which is possible by Fredholm alternative theorem since $U(\Theta) \perp \mathrm{Ker}\mathbb{L}_{S^n_+}$.

- Moreover

$$\bar{\omega}^p = \frac{1}{n}\int_{S^n_+} U(\Theta)\, d\sigma. \tag{29}$$

Fixed point argument

We want to find $\hat{\omega}^{p,r}$ and a vector $\Upsilon_{p,r} \in T_p\partial\Omega$ such that

$$\begin{aligned} rH(p,r,r\bar{\omega}^p + \hat{\omega}) &= n && \text{in } \Sigma_{p,r,r\bar{\omega}^p+\hat{\omega}}; \\ \langle N^{\Sigma}_{\partial\Sigma_{p,r,r\bar{\omega}^p+\hat{\omega}}}, N^{\partial\Omega}_{\partial\Sigma_{p,r,r\bar{\omega}^p+\hat{\omega}}} \rangle &= \langle \Upsilon, \tilde{\Theta} \rangle && \text{on } \partial\Sigma_{p,r,r\bar{\omega}^p+\hat{\omega}}. \end{aligned} \tag{30}$$

- Denote by Π the L^2 projection on $\operatorname{Ker} \mathbb{L}_{S^n_+} = span\{\Theta^1; \dots ; \Theta^n\}$, we have that

  $$\mathbb{L}_{S^n_+} : \Pi^\perp \mathcal{C}^{2,\alpha}(\bar{S^n_+}) \to \Pi^\perp \mathcal{C}^{0,\alpha}(\bar{S^n_+}) \tag{31}$$

  is invertible.
- Identifying $\operatorname{Ker} \mathbb{L}_{S^n_+}$ with $T\partial\Omega$, by a standard fixed point theorem, one can find a unique

  $$(\hat{\omega}^{p,r}, \Upsilon_{p,r}) \in \Pi^\perp \mathcal{C}^{2,\alpha}(\bar{S^n_+}) \times T_p\partial\Omega \tag{32}$$

  in a ball of radius Cr^2 solving (30).

- The fixed point argument yields a hypersurface $\Sigma_{p,\underline{r},r\bar{\omega}^p+\hat{\omega}^{p,r}} =: \Sigma_{p,r}$ which is $\mathcal{C}^{2,\alpha}$ close to S^n_+ and $\mathcal{C}^{1,\alpha}$ close to $\bar{\Sigma}_{p,r}$ we may assume that $\Sigma_{p,r}$ is embedded into Ω if r is small.
- Furthermore it satisfies

$$\begin{aligned} H_{\Sigma_{p,r}} &= \frac{n}{r} && \text{in } \Sigma_{p,r}; \\ \partial\Sigma_{p,r} &\subset \partial\Omega; && \\ \langle N^{\Sigma}_{\partial\Sigma_{p,r}}, N^{\partial\Omega}_{\partial\Sigma_{p,r}} \rangle &= \langle \Upsilon_{p,r}, \tilde{\Theta} \rangle && \text{on } \partial\Sigma_{p,r}. \end{aligned} \tag{33}$$

- We define the constraint functional on $\partial\Omega$ by

$$\varphi(p) = \mathcal{P}(E_{p,r}, \Omega) - \frac{n}{r} |E_{p,r}|, \tag{34}$$

where $E_{p,r}$ is the set bounded by $\Sigma_{p,r}$ and $\partial\Omega$.
Our Goal now is to show that $\varphi'(p_0) = 0 \Rightarrow \Upsilon_{p_0,r} = 0$.

Variational argument

If $q := exp_p^{\partial\Omega}(t\Upsilon)$, then for t sufficiently small the surface $\partial\Sigma_{q,r}$ is a graph over $\partial\Sigma_{p,r}$ for some smooth function $w_{p,r,\Upsilon,t}$ satisfying

$$\zeta_{p,r,\Upsilon} := \left(\frac{\partial w_{p,r,\Upsilon,t}}{\partial t}\Big|_{t=0} \right) N^{\partial\Omega}_{\partial\Sigma_{p,r}} \qquad \text{on } \partial\Sigma_{p,r} \subset \partial\Omega. \qquad (35)$$

Suppose that p is a critical point of φ then the first variation of area and volume yields

$$\begin{aligned} 0 &= d\varphi(p)[\Upsilon] \\ &= \int_{\Sigma_{p,r}} \left(H_{\Sigma_{p,r}} - \frac{n}{r} \right) \langle \zeta_{p,r,\Upsilon}, N_{\Sigma_{p,r}} \rangle d\sigma + \oint_{\partial\Sigma_{p,r}} \langle \zeta_{p,r,\Upsilon}, N^{\Sigma}_{\partial\Sigma_{p,r}} \rangle ds \end{aligned}$$

we conclude that

$$\oint_{\partial\Sigma_{p,r}} \langle \zeta_{p,r,\Upsilon}, N^{\partial\Omega}_{\partial\Sigma_{p,r}} \rangle \langle \Upsilon_{p,r}, \tilde{\Theta} \rangle ds = 0 \quad \forall \Upsilon \in T_p\partial\Omega. \qquad (36)$$

From the expansion of the metric and normals of $\Sigma_{p,r}$ we can deduce

$$|\langle \zeta_{p,r,\Upsilon}, N^{\partial\Omega}_{\partial\Sigma_{p,r}} \rangle + \langle \Upsilon, \tilde{\Theta} \rangle| \leq cr\|\Upsilon\|. \tag{37}$$

This implies that

$$\oint_{\partial\Sigma_{p,r}} \langle \Upsilon_{p,r}, \tilde{\Theta} \rangle \langle \Upsilon, \tilde{\Theta} \rangle ds \leq cr\|\Upsilon\| \oint_{\partial\Sigma_{p,r}} \langle \Upsilon_{p,r}, \tilde{\Theta} \rangle ds. \tag{38}$$

Using the expansion of the metric of small perturbed geodesic balls in $\partial\Omega$ we find that

$$\frac{1}{2}\mathrm{Area}(S^{n-1})r^{n-1}\|\Upsilon\|^2 \leq n \oint_{\partial\Sigma_{p,r}} |\langle \Upsilon, \tilde{\Theta} \rangle|^2 ds. \tag{39}$$

And, finally setting $\Upsilon = \Upsilon_{p,r}$ and using Hölder inequality we obtain

$$\oint_{\partial\Sigma_{p,r}} |\langle \Upsilon_{p,r}, \tilde{\Theta} \rangle|^2 ds \leq cr^2 \oint_{\partial\Sigma_{p,r}} |\langle \Upsilon_{p,r}, \tilde{\Theta} \rangle|^2. \tag{40}$$

Consequently there must be $\Upsilon_{p,r} = 0$ for r small.

The area and volume expansions of $\Sigma_{p,r}$ yields

$$\begin{aligned} r^{-n}\mathcal{P}(E_{p,r},\Omega) &= \mathcal{P}(B^{n+1},\mathbb{R}^{n+1}_{+}) + r(n+2)\int_{S^n_+}\langle B_{\partial\Omega}(p)\tilde{\Theta},\tilde{\Theta}\rangle\Theta^{n+1} \\ &\quad + O(r^2); \end{aligned}$$

$$\begin{aligned} r^{-1-n}|E_{p,r}| &= \frac{1}{n+1}\mathcal{P}(B^{n+1},\mathbb{R}^{n+1}_{+}) + \frac{r(n+3)}{n}\int_{S^n_+}\langle B_{\partial\Omega}(p)\tilde{\Theta},\tilde{\Theta}\rangle \\ &\quad - \frac{r}{n(n+2)}\langle B_{\partial\Omega}(p)E_i,E_i\rangle\int_{S^n_+}\Theta^{n+1}d\sigma + O(r^2), \end{aligned}$$

where $B_{\partial\Omega}$ is the second fundamental form of $\partial\Omega$. Hence

$$r^{-n}\varphi(p) = \frac{1}{n+1}\mathcal{P}(B^{n+1},\mathbb{R}^{n+1}_{+}) - c^1_n r H_{\partial\Omega}(p) + O(r^2) \qquad (41)$$

with

$$c^1_n = \int_{S^n_+}\left(\frac{2}{n+2} - (\Theta^i)^2\right)\Theta^{n+1}d\sigma, \qquad (42)$$

Setting

$$f(r,p) := \frac{1}{r}\left(r^{-n}\varphi(p) - \frac{1}{n+1}\mathcal{P}(B^{n+1},\mathbb{R}^{n+1}_{+})\right) \quad (43)$$

$$= -c_n^1 H_{\partial\Omega}(p) + O(r), \quad (44)$$

we have proved the following

Theorem

There exist $r_0 > 0$ and a smooth function $f : (0, r_0) \times \partial\Omega \to \mathbb{R}$ such that for every $r \in (0, r_0)$, if p is a critical point of $f(r,\cdot)$ then $\bar{\Sigma}_{p,r}$ can be perturbed to a smooth Capillary hypersurface $\Sigma_{p,r}$ with contact angle $\gamma = \frac{\pi}{2}$. Furthermore

$$\|f(r,\cdot) + c_n^1\, H_{\partial\Omega}\|_{C^1} \le c\, r, \quad (45)$$

A result similar to ours was first obtained by Ye [15] in the case where Ω is a compact manifold (without boundary) and partially generalized by Pacard and Xu [12]. It turns out that critical points of the scalar curvature of Ω determine the location for existence of CMC hypersurfaces.

As we observe here if Ω has a boundary, the mean curvature of $\partial\Omega$ is more relevant.

We emphasize that this result generalizes also to any constant contact angle $\gamma \in (0, \pi)$.

Concentrations at higher dimensional sets

If K is a k-dimensional smooth submanifold of $\partial\Omega$, we let $n := m - k$. Consider the "half"-geodesic tube:

$$\bar{\Sigma}_{K,r} := \{q \in \bar{\Omega} : \quad d(q,K) = r\}. \tag{46}$$

Notice that $\bar{\Sigma}_{K,r}$ satisfies almost the E-L for the Capillary problem with

$$\begin{aligned} r\, H_{\bar{\Sigma}_{K,r}} &= n + \mathcal{O}(r) && \text{in } \bar{\Sigma}_{K,r}, \\ \partial\bar{\Sigma}_{K,r} &\subset \partial\Omega, && \\ \langle N_{\bar{\Sigma}_{K,r}}, N_{\partial\Omega}\rangle &= 0 && \text{on } \partial\bar{\Sigma}_{K,r}. \end{aligned} \tag{47}$$

Our goal is to perturb $\bar{\Sigma}_{K,r}$ to a set close to it and satisfying our Geometric problem.

Perturbed tube

- All surfaces nearby K in $\partial\Omega$ can be parameterized by a parametric function: $\Phi : K \to NK^{\partial\Omega}$ in the following way:

$$K \ni p \longrightarrow \exp_p^{\partial\Omega}(\Phi) \tag{48}$$

- All surfaces nearby $\bar{\Sigma}_{K,r}$ in Ω can be parameterized by two parametric function:

$$\omega : K \times S^n_+ \to \mathbb{R} \quad \text{and } \Phi : K \to NK^{\partial\Omega} \tag{49}$$

in the following way:

$$S^n_+ \times K \ni (\Theta, p) \to \exp_p^{\partial\Omega}(r(1+\omega)\tilde{\Theta} + \Phi) + (r(1+\omega)\Theta^{n+1})N_{\partial\Omega} \tag{50}$$

- We will call $\Sigma(r, \omega, \Phi)$ the image of this map. Note that in particular

$$\partial\Sigma(r,\omega,\Phi) \subset \partial\Omega \qquad \text{and} \qquad \Sigma(r,0,0) = \bar{\Sigma}_{K,r}. \tag{51}$$

Assume that K is a minimal submanifold of $\partial\Omega$, then the mean curvature of $\Sigma(r,\omega,\Phi)$ can be expanded as

$$\begin{aligned} rm\,H(r,\omega,\Phi) &= n + rU(\Theta) + \mathcal{O}(r^2) \\ &- \mathcal{L}_r\,\omega - r\,\langle \mathfrak{J}\,\Phi, \tilde{\Theta}\rangle + r\mathcal{L}\omega + r\mathcal{J}(\Phi,\tilde{\Theta}) + \mathcal{Q}(\Phi,\tilde{\Theta},\tilde{\Theta}) \\ &+ r^2\,L(\omega,\Phi) + Q(\omega) + r\,Q(\omega,\Phi) \end{aligned}$$

and

$$\langle N_\Sigma, N_{\partial\Omega}\rangle = -\frac{\partial\omega}{\partial\eta} + r^2 L(\omega) + rQ(\omega) \quad \text{on } \partial\Sigma(r,\omega,\Phi), \qquad (52)$$

where

- $U(\Theta) \perp \mathrm{Ker}\,\mathbb{L}_{S^n_+}$
- $\mathcal{L}_r = r^2\Delta_K\omega + \Delta_{S^n_+}\omega + n\omega$
- $\langle \mathfrak{J}(\cdot), \tilde{\Theta}\rangle \in \mathrm{Ker}\,\mathbb{L}_{S^n_+}$ is the Jacobi operator of K while $\mathcal{J}(\Phi,\tilde{\Theta}) \perp \mathrm{Ker}\,\mathbb{L}_{S^n_+}$ is a linear map in Φ.

Let us analyze the main operator appearing in the mean curvature expansions:
for any $\omega \in \Pi^{\perp} L^2(S^n_+ \times K)$ $\quad \langle \Phi, \tilde{\Theta} \rangle \in \Pi L^2(S^n_+ \times K)$, let

$$\mathbb{L}_r(\omega, \Phi) := r^2 \Delta_K \omega + \Delta_{S^n_+} \omega + n\omega + \langle \mathfrak{J}\Phi, \tilde{\Theta} \rangle \tag{53}$$

so that the tube $\Sigma(r, \omega, \Phi)$ satisfies

$$\begin{aligned} r\, H(r, \omega, \Phi) &= \tfrac{n}{m} \\ \langle N_\Sigma, N_{\partial\Omega} \rangle &= 0 \end{aligned} \tag{54}$$

is equivalent to solve a non-linear PDE in the form:

$$\mathbb{L}_r(\omega, \Phi) + rL(\omega, \Phi) = rU(\Theta) + O(r^2) + Q(\omega, \Phi). \tag{55}$$

The non-linear PDE we want to solve is:

$$T_r(\omega, \Phi) := \mathbb{L}_r(\omega, \Phi) + rL(\omega, \Phi) = rU(\Theta) + O(r^2) + Q(\omega, \Phi) \tag{56}$$

We want to do this by a fixed point argument in the following way:

- if T_r is invertible we have a fixed point problem:

$$(\omega, \Phi) = -(T_r)^{-1}\left\{rU(\Theta) + O(r^2) + Q(\omega, \Phi)\right\} := F_r(\omega, \Phi). \tag{57}$$

- Since (ω, Φ) are assumed to be small perturbations, we need F_r to be defined from a ball $B(0, \delta(r))$ into itself for some $\delta(r) \to 0$ as $r \to 0$.
  So we have to estimate $\|(T_r)^{-1}\|$ which again has to be controlled by the error $rU(\Theta) + O(r^2)$.
- These two facts are what we are going to check now.

The spectrum of $\mathbb{L}_r$ is the union of

$$\{\Lambda_{ij} = \lambda_i + \frac{1}{r^2}(\mu_j - n)\}$$

and the spectrum of $\mathfrak{J}$.
Here $0 \le \lambda_i$ and $0 \le \mu_j$ are respectively the eigenvalues of Δ_K and $\Delta_{S^n_+}$ (with zero Neumann).
• K non-degenerate $\Longrightarrow 0 \notin \mathit{spectrum}(\mathfrak{J})$.
• A new difficulty arises since

$$\Lambda_{i0} := \lambda_i - \frac{n}{r^2} = 0 \qquad \text{when} \qquad r = \sqrt{\frac{n}{\lambda_i}}.$$

Nevertheless when $r \notin \{\sqrt{\frac{n}{\lambda_i}}, \quad , i \ge 1\}$ formal estimates of $\|\mathbb{L}_r\|^{-1}$ show that the error $rU(\Theta) + O(r^2)$ is too large if we want to apply a fixed point argument. Hence we need to improve it.

For that, letting $i \geq 1$ be an integer and setting

$$\omega_i = \sum_d^i r^d \omega_d \quad \text{and} \quad \Phi_i = \sum_d^{i-1} r^d \Phi_d,$$

- we have to solve

$$\begin{aligned} m\, r\, H(\omega_i, \Phi_i) &= n + \mathcal{O}(r^{i+1}) \quad \text{in} \quad \Sigma(\omega_i, \Phi_i), \\ \langle N_\Sigma, N_{\partial\Omega} \rangle &= \bar{\mathcal{O}}(r^{i+2}) \quad \text{on} \quad \partial\Sigma(\omega_i, \Phi_i). \end{aligned} \tag{58}$$

- This is equivalent to the iterative scheme:

$$\begin{aligned} \mathbb{L}_{S^n_+} \omega_d + \langle \mathfrak{J}\Phi_d, \tilde{\Theta} \rangle &= rU(\Theta) + \mathcal{O}(r^2) + rL(\omega_{d-1}, \Phi_{d-1}) \\ &\quad - r^2 \Delta\omega_{d-1} + Q(\omega_{d-1}, \Phi_{d-1}). \end{aligned} \tag{59}$$

- We can achieve (58) if K is non-degenerate buy noticing that $\langle \mathfrak{J}\Phi, \tilde{\Theta} \rangle$ is invertible and is acting on the Kernel of $\mathbb{L}_{S^n_+}$.

Now we estimate the distance from 0 to the spectrum of the selfadjoint Jacobi operator about $\Sigma(\omega_i, \Phi_i)$:

$$\mathbb{L}_{r,i}(\omega, \Phi) := r^2 \Delta\omega + \mathbb{L}_{S^n_+}\omega + \langle \mathfrak{J}\Phi, \tilde{\Theta} \rangle + rL_i(\omega, \Phi). \tag{60}$$

Following the idea of Malchiodi Montenegro [10], we want quantitative estimates of the inverse of this operator. For that we have to:

∗ Estimate the number of negative eigenvalues using the Weyl's asymptotic formula:

$$\sharp\{j \in \mathbb{N} \, : \, \lambda_j \leq \lambda\} \sim c_K \, \lambda^{\frac{k}{2}}. \tag{61}$$

∗ Estimate the derivative of its small eigenvalues $\lambda(r)$: $\frac{d\lambda(r)}{dr}$ by means of Kato's theorem.

We have

- The Morse Index (it is decreasing):

$$N_r := Ind\, \mathbb{L}_{r,i} \sim cr^{-k} \tag{62}$$

- The small eigenvalues $\lambda(r)$ are strictly monotone:

$$\frac{\partial \lambda(r)}{\partial r} = \frac{1}{r}(2n - o_r(1)). \tag{63}$$

- Hence letting $r_\ell = 2^{-\ell}$ the number of eigenvalues which cross 0, when r decreases from r_ℓ to $r_{\ell+1}$ is

$$N_{r_{\ell+1}} - N_{r_\ell} \simeq cr_\ell^{-k} \tag{64}$$

- From this we deduce that the set

$$B_\ell := \{r \in (r_{\ell+1}, r_\ell) \quad : \quad \mathrm{Ker}\, \mathbb{L}_r = \emptyset\} \tag{65}$$

contains an interval I_ℓ with $|I_\ell| \geq r_\ell^{k+1}$. Moreover for any $r \in I_\ell$, we have

$$\|\mathbb{L}_{r,i}^{-1}\|_{H^1} \leq cr^{-k}. \qquad \forall i \geq 1.$$

The fixed point problem can now be solved yielding the following result:

Theorem

Let Ω be a smooth bounded domain of $\mathbb{R}^{m+1}$, $m \geq 2$. Suppose that K is a non-degenerate minimal submanifold of $\partial\Omega$. Then, there exist a sequence $r_\ell \searrow 0$ such that the "half" geodesic tube $\bar{\Sigma}_{r_\ell,K}$ may be perturbed to a Capillary hypersurface $\Sigma_{r_\ell,K}$ with contact angle $\gamma = \frac{\pi}{2}$ and mean curvature $H_{\Sigma_{r_\ell,K}} \equiv \frac{n}{m}$ if ℓ is sufficiently large.

A result similar to ours was obtained by M-Mazzeo-Pacard [9] in the case where Ω is a compact manifold (without boundary). We have concentrations along non-degenerate minimal submanifold in Ω.

The minimality conditions is somehow necessary for the existence, as shown by Mazzeo-Pacard [11].

We emphasize that this result generalizes also to any constant contact angle $\gamma \in (0, \pi)$.

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