The Willmore Functional:
a perturbative approach

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-$(N, \mathring{g}) \hookrightarrow M$ isometrically immersed closed oriented surface
- H mean curvature, $H := A_{ij}\mathring{g}^{ij} = k_1 + k_2$
- The Willmore functional I is defined as

$$I(N) := \int H^2 d\Sigma$$

Applications

- Reilly Theorem:

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- Biology: Hellfrich Energy
- String Theory: Polyakov extrinsic action

Problem: find critical points

- $(N, \mathring{g}) \hookrightarrow (M, g)$ is a critical point of I if for all $\phi \in C^\infty(N)$, called ν the unit normal vector to N and

$$N(t) := exp_N(t\phi\nu) \quad \text{“} = N + t\phi\nu\text{”}$$

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- Critical points: called “Willmore Surfaces"
- N is a Willmore surface if and only if satisfies

$$2\triangle_N H + H\left(H^2 - 4k_1k_2 + 2Ric_M(\nu, \nu)\right) = 0$$

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- Divergence structure of the PDE: Riviére (2008)

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- GOAL: Say something about existence of Willmore surfaces in (non constantly) curved manifold

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USE A PERTURBATIVE METHOD

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AIM: find critical points of I_ϵ i.e. find $u \in H$ s.t

$$I_\epsilon'(u) = 0.$$

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CRUCIAL FACT: there exist a function in *finitely many* variables called *reduced fuctional*

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$$\{\text{critical points of } I_\epsilon|_{V_c}\} \leftrightarrow \{\text{critical points of } \Phi_\epsilon|Z_c\}$$

bijection

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$\Rightarrow$ to study the reduced functional Φ_ϵ is enough to know the $I_\epsilon|_Z$

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-Consider $\mathbb{R}^3$ with the metric $g_\epsilon = \delta + \epsilon h$, h a symmetric bilinear form

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-$I_0 =$ functional in euclidean metric
-Critical manifold
$Z = \{S_p^\rho : p \in \mathbb{R}^3, \rho > 0\} \simeq \mathbb{R}^3 \oplus \mathbb{R}^+$

Perturbed spheres

-look for critical points of the form $S^{\rho}_{p}(w)$, parametrized by

$$\Theta \mapsto p + \rho(1 - w(\Theta))\Theta$$

with $w \in C^{4,\alpha}(S^2)$.

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$\Rightarrow I_0$ satisfies the assumptions of the abstract method
AIM: give hypothesys on h to get that $\exists S_p^\rho(w)$ critical point.

Expansions on standard spheres

-recall $\Phi_\epsilon(S_p^\rho) = I_\epsilon(S_p^\rho) + o(\epsilon)$
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Lemma The following expansion for Φ_ϵ in ϵ holds

$$\begin{aligned}\Phi_\epsilon(S_p^\rho) &= 16\pi + 2\epsilon \int_{S^2} \Big[\mathrm{Tr}h - 3h(\Theta,\Theta) \\ &\quad +\rho A_{\mu\mu\lambda}\Theta^\lambda - \rho A_{\mu\nu\lambda}\Theta^\mu\Theta^\nu\Theta^\lambda\Big] d\Sigma_0 + o(\epsilon)\end{aligned}$$

with $A_{\mu\nu\lambda} := [D_\mu h_{\lambda\nu} + D_\lambda h_{\nu\mu} - D_\nu h_{\mu\lambda}]$.

Expansion for small radius

Lemma For small radius spheres

$$\Phi_\epsilon(S_p^\rho) = 16\pi - \frac{8\pi}{3} R_1(p)\rho^2\epsilon + o(\rho^2)\epsilon + o(\epsilon)$$

where $R_1 := \sum_{\mu\nu} D_{\mu\nu} h_{\mu\nu} - \triangle \mathrm{Tr} h$.

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Remark The scalar curvature R_{g_ϵ} of g_ϵ is

$$R_{g_\epsilon} = \epsilon R_1 + o(\epsilon)$$

Existence Theorem

Theorem Assume

- $\exists \bar{p} \in \mathbb{R}^3$ such that $R_1(\bar{p}) \neq 0$,

- Said $\|h(p)\| := \sup_{|v|=1} |h_p(v, v)|$

$i)$ $\lim_{|p|\to\infty} \|h(p)\| = 0$.

$ii)$ $\exists C > 0$ and $\alpha > 2$ s.t.

$$|D_\lambda h_{\mu\nu}(p)| < \frac{C}{|p|^\alpha} \quad \forall \lambda, \mu, \nu = 1 \ldots 3.$$

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Then, for ϵ small enough, there exist $(p_\epsilon, \rho_\epsilon) \in \mathbb{R}^3 \oplus \mathbb{R}^+ +$ and $w_\epsilon(p_\epsilon, \rho_\epsilon) \in C^{4,\alpha}(S^2)^\perp$ s.t the perturbed sphere $S^{\rho_\epsilon}_{p_\epsilon}(w_\epsilon(p_\epsilon, \rho_\epsilon))$ is critical point of the Willmore functional I_ϵ.

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-fixed $p \in M$, the *geodesic sphere* $S_{p,\rho}$ of radius ρ is the image of $\Theta \in S^2 \subset T_pM \mapsto Exp_p(\rho\Theta)$
- fix a function $w \in C^{4,\alpha}(S^2)$, the *perturbed geodesic sphere* $S_{p,\rho}(w)$ is defined as the image of

$$\Theta \in S^2 \subset T_pM \mapsto Exp_p\big(\rho(1 - w(\Theta))\Theta\big)$$

Expansion of the Willmore functional

The Willmore functional $\int H^2 d\Sigma$ on $S_{p,\rho}(w)$ can be expanded as

$$I(S_{p,\rho}(w)) = 16\pi - \frac{8\pi}{3}R(p)\rho^2 + \int_{S^2}(Q_p^{(2)}(w) + \rho^2 L_p(w))d\Theta + O_p(\rho^3),$$

where R is the scalar curvature of the ambient manifold (M, g).

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Lemma For small ρ and w, if $S_{p,\rho}(w)$ is a critical point then

$i)$ w is a C^1 function in (p,ρ),
$ii)$ $\|w\| = O(\rho^2)$ for $\rho \to 0$,
$iii)$ $\frac{\partial}{\partial\rho} w(p,\rho)$ is bounded as $\rho \to 0$

$$\Longrightarrow \frac{\partial}{\partial\rho} I[S_{p,\rho}\big(w(p,\rho)\big)] \; = \; -\frac{16\pi}{3} R(\bar{p})\rho + O(\rho^2)$$
$$\neq \; 0 \text{ if } R(\bar{p}) \neq 0 \text{ and } \rho << 1$$

Non Existence

THEOREM: Assume that at the point $\bar{p} \in M$ the scalar curvature is non null:

$$R(\bar{p}) \neq 0.$$

Then, for radius ρ and perturbation $w \in C^{4,\alpha}(S^2)$ small enough, the surfaces $S_{\bar{p},\rho}(w)$ are not critical points of the Willmore functional I.

References

-About the method: A. Ambrosetti, A. Malchiodi, “Perturbation methods and semilinear elliptic problems in $\mathbb{R}^n$", Progress in mathematics, Birkhauser (2006).

-The results: A. Mondino, “Some results about the existence of critical points for the Willmore functional", Math. Zeit. (2009, in press).

Thank you