Mirror couplings of reflecting Brownian motions and applications

Mihai N. Pascu

Transilvania University of Braşov, Romania

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Abstract

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In the present talk we will show that the coupling can be extended to the case when the two reflecting Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

As an application of the construction, we will derive a unifying proof of the two most important results on Chavel's conjecture on the domain monotonicity of the Neumann heat kernell ([Ch], [Ke]).

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A d-dimensional Brownian motion starting at $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ is a stochastic process $B_t = (B_t^1, \ldots, B_t^d)$, where the components B_t^i are independent 1-dimensional Brownian motions starting at x^i, $1 \leq i \leq d$.

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Reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ starting at $x_0 \in \bar{D}$ is a solution of the stochastic differential equation

$$X_t = x_0 + B_t + \int_0^t \nu_D(X_s)\, dL_s^X, \qquad t \geq 0, \tag{1}$$

where B_t is a d-dimensional BM starting at $B_0 = 0$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, L_t^X is the local time of X on the boundary of D, X_t is $\mathcal{F}_t$-adapted and almost surely $X_t \in \overline{D}$ for all $t \geq 0$.

Skorokhod map

Remark

It can be shown ([LiSz]) that there exists a unique $\mathcal{F}_t$-semimartingale which solves (1). In fact, there exists a map (Skorokhod map)

$$\Gamma : C\left([0,\infty) : \mathbb{R}^d\right) \to C\left([0,\infty) : \bar{D}\right)$$

such that $X = \Gamma(x+B)$ a.s.
For each $T>0$ fixed, $\Gamma|_{[0,T]}$ is Hölder continuous of order $1/2$ on compact subsets of $C\left([0,T] : \mathbb{R}^d\right)$.

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Mirror coupling of Brownian motions

Given a hyperplane $\mathcal{H}$ (*the mirror*) and a Brownian motion X_t, we define the Brownian motion Y_t as the mirror image of X_t with respect to $\mathcal{H}$ until the coupling time

$$\xi = \inf\{s > 0 : X_s = Y_s\},$$

after which the processes X_t and Y_t evolve together.

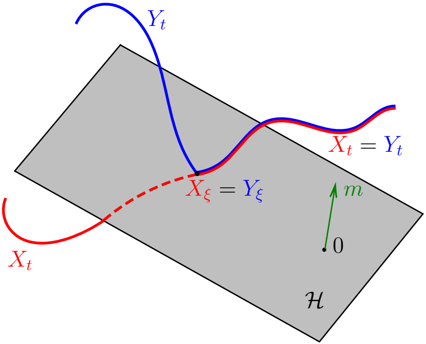

Figure: The mirror coupling of Brownian motions.

Equation of the mirror coupling of Brownian motions

If m is a unit normal to $\mathcal{H}$, then Y_t is given explicitly by

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Introducing the $d \times d$ matrix H by

$$H\left(m\right) = I - 2m\, m^T = \left(\delta_{ij} - 2m_i m_j\right)_{1 \leq i,j \leq d}, \tag{3}$$

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$$Y_t = G\left(Y_t - X_t\right) X_t, \qquad t \geq 0, \tag{4}$$

where

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$$X_t = x + W_t + \int_0^t \nu_{D_1}(X_s)\, dL_s^X \qquad (6)$$

$$Y_t = y + Z_t + \int_0^t \nu_{D_2}(Y_s)\, dL_s^Y \qquad (7)$$

$$Z_t = \int_0^t G(Y_s - X_s)\, dW_s \qquad (8)$$

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Considering Γ and $\tilde{\Gamma}$ the corresponding Skorokhod maps (i.e. $X = \Gamma(x + W)$,
$Y = \tilde{\Gamma}(y + Z)$), the above system is equivalent to

$$Z_t = \int_0^t G\left(\tilde{\Gamma}(y + Z)_s - \Gamma(x + W)_s\right) dW_s \tag{9}$$

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Remark

In the particular case when $D_1 = D_2$, (6) – (9) above reduces to the case considered by Burdzy et. al. (i.e. mirror coupling of reflecting Brownian motions in D).

Main result

Theorem

Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded domains with $\overline{D_2} \subset D_1$ and D_2 convex domain, and let $x \in \bar{D}_1$ and $y \in \bar{D}_2$ be arbitrarily fixed points.
Then there exists a strong solution X_t, Y_t to (6) – (9) above, referred to as a mirror coupling of reflecting Brownian motions in D_1, respectively D_2, starting from $(x, y) \in \overline{D_1} \times \overline{D_2}$ with driving Brownian motion W_t.

Some remarks

Remark

In the case $D_1 = D_2 = D$, the solution to (9) can be essentially constructed by Picard iterations, since outside of the origin G satisfies

$$\left|\left|G(u) - G\left(u'\right)\right|\right| \leq c\left|u - u'\right|,$$

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Remark

In the general case this method cannot be used. The reason is that once the processes X_t and Y_t have coupled, it is possible for them to decouple: for example if $X_t = Y_t \in \partial D_2$, the solutions will split.
The behaviour of G at the origin becomes therefore essential – we have to show the existence of a degenerate SDE (G is discontinuous at the origin).
Surprisingly, the existence of the solution comes from the convexity of the smaller domain!

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- Approximate $D_2 = D$ by an increasing sequence of convex polygonal domains $D_n \nearrow D$
- Show the solution Y_t^n for D_n converges to the solution Y_t for D, that is

$$Z_t^n = \int_0^t G\left(Y_s^n - X_s\right) dW_s \underset{n\to\infty}{\longrightarrow} \int_0^t G\left(Y_s - X_s\right) dW_s = Z_t, \qquad t \geq 0, \qquad (10)$$

where Z_t^n, Z_t are the driving Brownian motions for Y_t^n, respectively Y_t.

Applications

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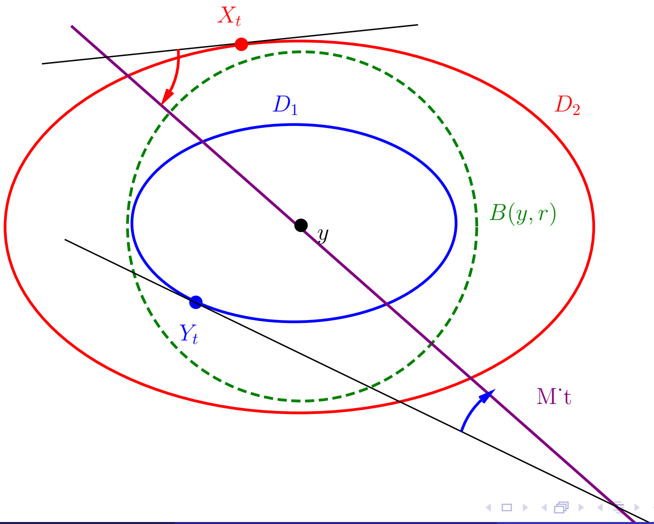
Wilfried Kendall proved the conjecture in the case when D_1 is a ball centered at x or y and D_2 is convex (coupling arguments).
Using the mirror coupling we can give a unifying proof of Chavel conjecture in the case

$$D_1 \subset B \subset D_2$$

where B is a ball centered at either x or y.

Geometry of the mirror coupling

Consider a mirror coupling (X_t, Y_t) of reflecting Brownian motions in (D_2, D_1) starting at $x \in D_1$.



The proof of Chavel conjecture

If $D_1 \subset B(y, r) \subset D_2$, then the mirror M_t of the coupling cannot separate Y_t and y:

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hence

$$p_{D_2}(t,x,y) = \lim_{\varepsilon \searrow 0} \frac{1}{|B(y,\varepsilon)|} P^x(X_t \in B(y,\varepsilon)) \leq \lim_{\varepsilon \searrow 0} \frac{1}{|B(y,\varepsilon)|} P^x(Y_t \in B(y,\varepsilon)) = p_{D_1}(t,x,y).$$

Extensions of the mirror coupling

Same arguments can be used in order to construct the mirror coupling in $D_1, D_2 \subset \mathbb{R}^d$ if:

- D_1 and D_2 have non-tangential boundaries (needed for localization of the construction)
- $D_1 \cap D_2$ is a convex domain (needed for the construction of the solution).

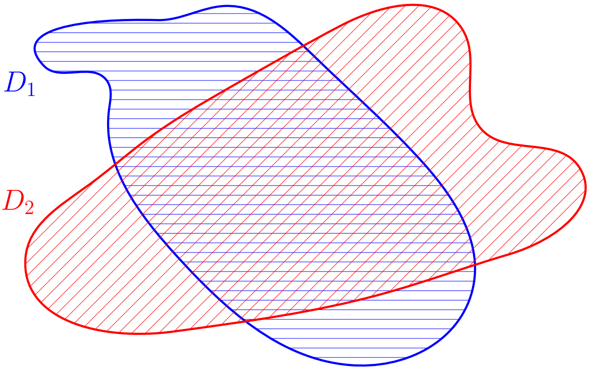

Figure: Generic smooth domains D_1, D_2 for the mirror coupling

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$$\sigma(u) = \begin{cases} 1, & u \neq 0 \\ 0, & u = 0 \end{cases}.$$

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The above has the solutions $U_t \equiv 0$ and $U_t = W_t$, and a whole range of intermediate solutions (sticky Brownian motion).

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The above has the solutions $U_t \equiv 0$ and $U_t = W_t$, and a whole range of intermediate solutions (sticky Brownian motion).
The original equation has solutions $Y_t = X_t = W_t$ (sticky mirror coupling), $Y_t = -X_t = -W_t$ (non-sticky mirror coupling), and a whole range of intermediate solutions (weak/mild sticky mirror coupling).

Question of uniqueness

The solution is not unique.
In the case $D_1 = D_2 = \mathbb{R}$, with the substitution $U_t = -\frac{Y_t - X_t}{2}$, we obtain the singular SDE:

$$U_t = \int_0^t \sigma(U_s)\, dW_s, \tag{11}$$

where

$$\sigma(u) = \begin{cases} 1, & u \neq 0 \\ 0, & u = 0 \end{cases}.$$

The above has the solutions $U_t \equiv 0$ and $U_t = W_t$, and a whole range of intermediate solutions (sticky Brownian motion).
The original equation has solutions $Y_t = X_t = W_t$ (sticky mirror coupling), $Y_t = -X_t = -W_t$ (non-sticky mirror coupling), and a whole range of intermediate solutions (weak/mild sticky mirror coupling).

References

- R. Atar and K. Burdzy, On Neumann eigenfunctions in lip domains, J. Amer. Math. Soc. **17** (2004) , pp. 243-265.
- R. Atar, K. Burdzy, Mirror couplings and Neumann eigenfunctions, Indiana Univ. Math. J. **57** (2008), No. 3, pp. 1317 – 1351.
- R. Bañuelos and K. Burdzy, On the "Hot Spots" conjecture of J. Rauch, J. of Funct. Ann. **164** (1999), pp. 1 – 33 .
- R. Bañuelos, T. Kulczycki, B. Siudeja, Neumann heat kernel monotonicity, Potential Anal. **30** (2009), No. 1, pp. 65 – 83.
- K. Burdzy, Krzysztof, Neumann eigenfunctions and Brownian couplings, Potential theory in Matsue, 11–23, Adv. Stud. Pure Math. **44**, Math. Soc. Japan, Tokyo, 2006.
- K. Burdzy and W. Kendall, Efficient Markovian couplings: Examples and counterexamples, Ann. Appl. Probab. **10** (2000), No. 2, pp. 362 – 409.
- B. Kawohl, Rearrangements and convexity of level sets in PDE, Lecture Notes in Mathematics, Vol. **1150**, Springer-Verlag (1985).
- I. Chavel, Heat diffusion in insulated convex domains, J. London Math. Soc. (2) **34** (1986), No. 3, pp. 473 – 478.
- W. S. Kendall, Coupled Brownian motions and partial domain monotonicity for the Neumann heat kernel, J. Funct. Anal. **86** (1989), No. 2, pp. 226 – 236.
- P. L. Lions, A. S. Sznitman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. **37** (1984), No. 4, pp. 511 – 537.