Integral properties of the warping function of multiple connected plane domain

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Let G be a multiple connected plane domain. The outer boundary curve of G will be denoted by Γ_0 and the internal boundary curves will be called $\Gamma_1, \ldots, \Gamma_n$. As it is known the warping function $u(x, G)$ of G (the stress function) satisfies

$$\begin{cases} \triangle u = -2 & \text{in } G, \\ u = 0 & \text{on } \Gamma_0, \\ u = c_i & \text{on } \Gamma_i, \ i = 1, \ldots, n. \end{cases} \tag{1}$$

The constants c_i are determined by the conditions

$$\oint_{\Gamma_i} \frac{\partial u}{\partial n}\, ds = -2a_i, \ i = 1, \ldots, n,$$

where $\partial/\partial n$ denotes the normal derivative on Γ_i, and a_i is the area enclosed by Γ_i.

A most important functional in the torsion problem of a long cylindrical beam of uniform cross section is the torsional rigidity $P(G)$. According to St Venant it may be defined as follows

$$P(G) := \int_G |\nabla u(x,G)|^2 \, dx. \tag{2}$$

Another example of a physical functional of a domain constructed with the help of the warping function is

$$u(G) := \sup_{x \in G} u(x,G). \tag{3}$$

The most famous and important inequality in the torsion problem is St Venant conjecture for simply connected domains, that was proved by G. Polya [8]. Today the isoperimetric inequality

$$P(G) \leq \frac{A(G)^2}{2\pi} \tag{4}$$

is known as the St Venant—Polya inequality, here $A(G)$ is the area of G. G. Polya and H.F. Weinstein extended the last inequality to multiply connected domains. They have proved the inequality

$$P(G) \leq \frac{1}{2\pi}\left[A_0^2 - \left(\sum_{i=1}^{n} a_i\right)^2\right], \tag{5}$$

where A_0 is the area inclosed by the curve Γ_0.

In the treatise "Isoperimetric Inequalities in Mathematical Physics" by G. Polya and G. Szegö a related isoperimetric inequality

$$2\pi u(G) \leq A(G) \tag{6}$$

was proved for simply connected domains, and it was shown by L.E. Payne [1], that the inequality remains valid for multiply connected domains. More over, in the case of simply connected domains the last inequality is a consequence of the St Venant—Polya inequality and the inequality

$$2\pi u(G)^2 \leq P(G), \tag{7}$$

which was obtained by L.E. Payne [1].

Also, L.E. Payne [1] have proved two inequalities

$$2\pi u(G)^2 \leq P(G) - \sum_{i=1}^{n} c_i a_i, \; P(G) - \sum_{i=1}^{n} c_i a_i \leq \frac{A(G)^2}{2\pi} \tag{8}$$

from which all the above mentioned inequalities are followed.

The main goal of our report is to establish, that all the mentioned inequalities are consequences of an integral property of the warping function. Our method is based on a conception of "isoperimetric monotonicity with respect to a parameter" in the Mathematical Physics. Using the warping function we construct a new physical functional, such that it satisfies an isoperimetric monotonicity property with respect to the parameter.

It should be mentioned that the first isoperimetric monotonicity property for simply connected domains was conjectured by J. Hersch and later it was proved by M.-Th. Kohler–Jobin [13]. It deals with a functional of the solution of boundary value problem, which depends on the parameter. The proved by M.-Th. Kohler–Jobin isoperimetric monotonicity property contains and generalizes several known isoperimetric inequalities as special cases, in particular, the St Venant — Polya inequality.

1 Main results and corollaries

Let

$$T_p(G) = \int_G u(x,G)^p \, dA. \tag{9}$$

With the help of Green's formulae one can get

$$P(G) = 2T_1(G) + 2\sum_{i=1}^{n} c_i a_i.$$

Let us introduce the following physical functional

$$\Phi(p) := \frac{1}{u(G)^{p+1}} \left(T_p(G) - \frac{2\pi u(G)^{p+1}}{p+1} \right), \tag{10}$$

where $p \geq 0$. L.E. Payne proved that, the excretion in the brackets is not negative, and its turns identically zero iff G is a concentric ring.

Theorem 1. *Let G is a multiply connected domain, and $T_{p_0}(G) < +\infty$ for some $p_0 \in [0, \infty)$. Then*
1) *If G is not a concentric ring, then $\Phi(p)$ is a strictly decreasing function of p for $p \geq p_0$.*
2) *If G is a concentric ring, then $\Phi(p) \equiv 0$, for $p \in [0, +\infty)$.*

From this theorem follows.

Corollary 1. *Let G is a multiply connected domain, and $T_q(G) < +\infty$ $(q \geq 0)$. If $p > q$, we have*

$$T_p(G) \leq u(G)^{p-q} T_q(G) - \frac{2\pi(p-q)u(G)^{p+1}}{(p+1)(q+1)}. \tag{11}$$

Equality holds if and only if G is a concentric ring.

The inequality (11) can be written as

$$\Phi(p) \leq \Phi(q) \quad (p > q). \tag{12}$$

Let us put $q = 0$ in (11), then

$$T_p(G) \leq u(G)^p A(G) - \frac{2\pi p u(G)^{p+1}}{p+1}.$$

If we put $p = 1$, then the last inequality becomes Payne's isoperimetric inequality (8), which has a number of remarkable consequences established by L.E. Payne [1].

It was proved by myself that the physical functional

$$\varphi(\alpha) := \left(\frac{\alpha}{2\pi}\int_G u^{\alpha-1}(x,G)\mathrm{d}A\right)^{1/\alpha} \tag{13}$$

for simply connected domains, and $\alpha \geq 1$ has an isoperimetric monotonicity property with respect to the parameter α.

Theorem 2. *If the assumptions of Theorem 1 hold, then:*

1) If G is not a concentric ring, then $\varphi(\alpha)$ is a strictly decreasing function for $\alpha > p_0$.

2) For a concentric ring R $\varphi(\alpha) \equiv \left(r_0^2 - r_1^2\right)/2$, where r_0, r_1 are radiuses of the boundary circles of R.

Thus the isoperimetric monotonicity property of the functional $\varphi(\alpha)$ remains valid for multiply connected domains.

Corollary 2. *Let G is a multiply connected domain, and $\|u(x,G)\|_q < +\infty$ $(q > 0)$. If $p > q$, we have*

$$\frac{\|u(x,G)\|_q}{\|u(x,G)\|_p} \geq \frac{\|u^*(x,R)\|_q}{\|u^*(x,R)\|_p}, \tag{14}$$

here the ring R has the same joint area of the holes as the domain G, and satisfies the condition

$$\|u^*(x,R)\|_t = \|u(x,G)\|_t, \tag{15}$$

where $q \leq t \leq p$. Equality holds if and only if G is a concentric ring.

Let us remark that we can get an inequality between different norms of $u(x,G)$ without additional normalization.

Corollary 3. *Under the assumptions of Theorem 2, we have*

$$\frac{(\|u(x,G)\|_q)^{q(p+1)}}{(\|u(x,G)\|_p)^{p(q+1)}} \geq \frac{(2\pi)^{p-q}(p+1)^{q+1}}{(q+1)^{p+1}}, \tag{16}$$

The equality sign holds if and only if G is a concentric ring.

The last assertion is equivalent to $\varphi(p+1) \leq \varphi(q+1)$, which is a direct consequence of Theorem 2.

Let us remark that the inequality $\varphi(2) \leq \varphi(1)$ is coincides with (8) obtained by Payne in [1]. Also, the inequality $\varphi(\infty) \leq \varphi(p)$ is equivalent to

$$T_p(G) \geq \frac{2\pi u(G)^{p+1}}{p+1} \tag{17}$$

already proved by Payne.

In the next two assertions we give estimates for a class of integrals of the warping function.

Let a function $F(t)$ has the representation

$$F(t) := p \int\limits_0^t s^{p-1} f(s) ds, \tag{18}$$

where $p > 0$, and $f(s)$ is another function, whose properties play an important role, as we see below.

Theorem 3. *Let G is a multiply connected domain and let $p > 0$ such that $T_p(G) < +\infty$. Then*
1) *If $f(s)$ is not a decreasing function, then*

$$\int_G F(u(x,G))\, dx \leq \int_{R_p} F(u(x,R_p))\, dx.$$

2) *If $f(s)$ is not a increasing function, then the inverse inequality holds*

$$\int_G F(u(x,G))\, dx \geq \int_{R_p} F(u(x,R_p))\, dx.$$

Here R_p is the concentric ring with the same joint area of the holes as on G, and the ring R_p satisfy the equality $T_p(R_p) = T_p(G)$. The both equalities hold if and only if G is a ring bounded by two concentric circles.

Using the functionals $T_p(G)$ and $u(G)$ we can get explicit bounds for integrals of the warping function.

Theorem 4. *Under the assumptions of Theorem 3 the following estimates hold:*

$$\int_G F(u(x,G))\,dx \leq$$

$$\leq \frac{T_p(G)}{u(G)^p} F(u(G)) - \frac{2\pi u(G) F(u(G))}{p+1} + 2\pi \int\limits_0^{u(G)} F(t)\,dt,$$

where $f(s)$ is not a decreasing function, and

$$\int_G F(u(x,G))\,dx \geq$$

$$\geq \frac{T_p(G)}{u(G)^p} F(u(G)) - \frac{2\pi u(G) F(u(G))}{p+1} + 2\pi \int\limits_0^{u(G)} F(t)\,dt,$$

here $f(s)$ is not a increasing function.

Equalities hold iff G is a concentric ring.

Indeed, we can get Theorem 1 from Theorem 4 with a special choice of $f(t)$. Also, we prove Theorem 2 with the help of Theorem 1. Note that Theorem 4 gives better estimates than Theorem 3.

This is the end of my talk.

THANK YOU VERY MUCH
FOR YOUR ATTENTION.

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