An isoholonomic approach to Riemannian Geometry

"From Carthage to the World"

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Outline

Riemannian Metrics

Definition

A Riemannian manifold is a smooth manifold together with a symmetric $(0,2)$-tensor g that at every tangent space T_pM is a positive definite inner product $\langle \cdot, \cdot \rangle$.

Parallel Transport

By way of the fundamental theorem of Riemannian Geometry, given a Riemannian Manifold (M, g) there exists a unique torsion free metric connection ∇ on TM compatible with g; i.e. that satisfies that for any vector fields X, Y, Z on M,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Parallel Transport

More over, if a one considers a curve with parameter t on M and a vector field X defined along such curve, there is a covariant derivative $\frac{D}{\partial t}X$ of X along the curve. Also satisfying that for any such vectors X, Y,

$$\frac{\partial}{\partial t}\langle X, Y\rangle = \langle \frac{D}{\partial t}X, Y\rangle + \langle X, \frac{D}{\partial t}Y\rangle$$

Remark

We will denote $\frac{D}{\partial t}X$ simply by X'.

Parallel Transport

Definition

The solutions to ODE given by

$$\begin{cases} X'(t) = 0 \\ X(0) = v \end{cases}$$

along a curve with parameter t are called *parallel vector fields*, and their images *parallel translate* of v. Clearly, by Leibniz rule, for any two parallel vector fields we have that

$$\langle X, Y \rangle' = 0,$$

and hence the name.

Geodesics

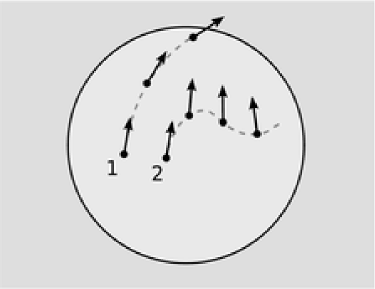

Figure: Parallel tangent field

Parallel Transport

For a given curve α on M, we will denote the parallel translation along α by

$$P_t^\alpha : T_{\alpha(s)}M \mapsto T_{\alpha(s+t)}M,$$

for s and $t+s$ in the domain of α.

It is well-known that these maps are linear isometries and that they don't depend on the parametrization of α. Because of this, we shall henceforward assume that all curves have domain equal to the interval $[0,1]$

Holonomy

If one considers only loops based at any given point of $p \in M$ then one has the following

Definition

The collection $\{P_1^\alpha\}$ over all loops α is a subgroup of the orthogonal group of the fiber $T_{\alpha(0)}M$ is called *holonomy group* of the connection at p, and is denoted by

$$\mathrm{Hol}_p = \mathrm{Hol}_p M = \mathrm{Hol}_p(g) = \mathrm{Hol}_p(\nabla).$$

On the 2-sphere

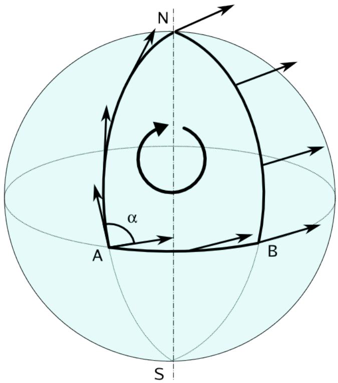

Figure: The holonomy group is all rotations

Group-norms

Definition

Let G be any group. A *group-norm* on G is a function $N : G \to \mathbb{R}$ that satisfies the following properties.

1. Positivity: $N(A) \geq 0$
2. Non-degeneracy: $N(A) = 0$ iff $A = id_V$
3. Symmetry: $N(A^{-1}) = N(A)$
4. Subadditivity ("Triangle inequality"): $N(AB) \leq N(A) + N(B)$.

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Example

Let V be a normed vector space and let G be a subgroup of the group of norm preserving automorphisms of V. Then $N(A) = \|id_V - A\|$ is a group-norm.

Group-norms

Proposition

A group G together with a group-norm N becomes a topological group with the left invariant metric induced by

$$d(A, B) = N(A^{-1}B). \tag{2.1}$$

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Proposition

With the N-topology on G, the group-norm N is continuous.

A holonomic space

Definition

Let $(V, \|\cdot\|)$ be a normed vector space, $H \leq Aut(V)$ a subgroup of norm preserving linear isomorphisms, and $L : H \to \mathbb{R}$ a group-norm on H. The triplet (V, H, L) will be called a *holonomic space* if it further satisfies the following convexity property:

(P) For all $u \in V$ there exists $r = r_u > 0$ such that for all $v, w \in V$ with $\|v - u\| < r$, $\|w - u\| < r$, and for all $A \in H$,

$$\|v - w\|^2 - \|v - Aw\|^2 \leq L^2(A). \tag{2.2}$$

A holonomic space

Lemma

Given a holonomic space (V, H, L) as above, there exists $r > 0$ such that for $u \in V$, $|u| < r$, and for any $B \in H$,

$$\|u - Bu\| \leq L(B). \tag{2.3}$$

A holonomic space

Theorem

Let (V, H, L) be a holonomic space.

$$d_L(u, v) = \inf_{a \in H} \left\{ \sqrt{L^2(a) + \|u - av\|^2} \right\}, \tag{2.4}$$

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Definition

Given a holonomic space (V, H, L). The metric given by (2.4) will be called *associated holonomic metric* and V together with this metric will be denoted by V_L.

A holonomic space

Theorem

A triplet (V, H, L) is a holonomic space if and only if $id : V \to V_L$ is a locally isometry.

A holonomic space

Definition

Let (V, H, L) be a holonomic space. The *holonomy radius* of a point $u \in V$ is the supremum of the radii $r > 0$ satisfying the convexity property (P) given by (2.2). It will be denoted by $\mathrm{HolRad}(u)$. It may be infinite.

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Remark

The holonomy radius is also the radius of the largest ball so that the restricted d_L-metric is Euclidean.

Introduction

The starting point for studying the metric geometric properties of bundles over Riemannian manifolds is to consider their total spaces as Riemannian manifolds such that the projection is a Riemannian submersion. Existence and naturality of such metrics has been addressed and extensively studied from a purely differential geometric viewpoint.

One procedure to view a vector bundle as a Riemannian submersion is to endow the base with a Riemannian metric and to require that the bundle be equipped with a bundle metric and any compatible bundle connection. These two ingredients provide a plethora of metrics on the total space of the bundle, perhaps the simplest of which is the Sasaki-type metric, introduced for the tangent bundle by Sasaki.

These constructions generalize

Remark

Given a vector bundle with metric and connection, (E, h, C, ∇), parallel translation is by isometries.

Definition

Given a bundle with metric and connection, parallel translation yields a map from the space of piecewise smooth loops at a point $p \in M$, Ω_p, to the group $GL(E_p)$ by

$$\alpha \in \Omega_p \mapsto H(\alpha) = P_1^\alpha. \tag{3.1}$$

The holonomy group Hol_p at the point p on the base manifold is then defined as the continuous image of H.

Vertical lift

Definition

Given a (normed) vector space V, there is a canonical isomorphism between $V \times V$ and TV, given by

$$\mathfrak{I}_v(w)f = \mathfrak{I}(v,w)f = \frac{d}{dt}\bigg|_{t=0} f(v+tw). \tag{3.2}$$

That is, $\mathfrak{I}_v w$ is the directional derivative at v in the direction w.

Remark

Given any vector bundle (E,π), (3.2) yields a bundle isomorphism between $\oplus^2 E := E \oplus E$ and the vertical distribution $\mathcal{V} = \ker \pi_* \subseteq TE$, in a natural way.

Horizontal lift

Proposition

A connection on (E, π, M) can be interpreted as a splitting C of the following short exact sequence of bundles over the the total space E.

$$0 \longrightarrow \pi^* E \xrightarrow{\mathfrak{I}} TE \underset{C}{\overset{\psi}{\rightleftarrows}} \pi^* TM \longrightarrow 0 \tag{3.3}$$

where $\psi = (\pi_E, \pi_*)$, by regarding $C(e, u)$ as the horizontal lift of the vector $x \in M_{\pi e}$ to e.

Summary

Given a section σ on E, the vertical lift σ^v is the vector field such that at any $f \in E$,

$$\sigma^v(f) := \mathfrak{I}_f(\sigma(\pi(f))).$$

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Given a vector field X on M, the horizontal lift X^h is the vector field such that at any $f \in E$,

$$X^h := C(f, X(\pi(f)))$$

Sasaki-type metrics

Definition

Given a vector bundle with metric and compatible connection (E, π, h, ∇^E) over a Riemannian manifold (M, g), the *Sasaki-type metric* $\mathrm{G} = \mathrm{G}(g, h, \nabla^E)$ is defined as follows

$$\mathrm{G}(e^v, f^v) = h(e, f), \quad (3.4)$$

$$\mathrm{G}(e^v, x^h) = 0, \quad (3.5)$$

$$\mathrm{G}(x^h, y^h) = g(x, y). \quad (3.6)$$

Riemannian

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Also, the metric on the fibers is totally geodesic and flat.

Metric

The aim is to determine the metric-space structure. Recall that for any Riemannian manifold, the length distance is given by the infimum of lengths over curves.

$$d(u, v) = \inf \ell(\gamma),$$

where γ is a curve from u to v.

Metric

Parallel transport along a curve gives a metric trivialization of the bundle along the curve, so that the metric on the restricted bundle is given by

$$\alpha^{*}\mathrm{G} = \ell(\alpha)^2 dt^2 + \alpha^{*} h_p,$$

where $p = \alpha(0)$, and ℓ denotes the length of α.

Result

Theorem

The length distance on (E, G) is expressed as follows. Let $u, v \in E$

$$d_E(u, v) = \inf \sqrt{\ell(\alpha)^2 + \|P_1^\alpha u - v\|^2}, \tag{4.1}$$

over all $\alpha : [0,1] \to M, \alpha(0) = \pi u, \alpha(1) = \pi$.
Furthermore, if $\pi u = \pi v$ then

$$d_E(u, v) = \inf\{\sqrt{L(a)^2 + \|au - v\|^2} : a \in Hol_p\}, \tag{4.2}$$

with L being the infimum of lengths of loops yielding a given holonomy element.

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with L being the infimum of lengths of loops yielding a given holonomy element.

An unexpected group norm

Theorem

Let Hol_p be the holonomy group over a point $p \in M$ of a bundle with metric and connection and suppose that M is Riemannian. Then the function $L_p : Hol_p \to \mathbb{R}$,

$$L_p(A) = \inf\{\ell(\alpha) | \alpha \in \Omega_p, P_1^{\alpha} = A\}, \tag{5.1}$$

is a group-norm for Hol_p.

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Proof (continued).

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$$d(u, Au) \leq \sqrt{L(A)^2 + \|Au - Au\|^2} = 0.$$

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A contradiction!

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Holonomy space revisited

Definition

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Theorem

Let E_p be the fiber of a vector bundle with metric and connection E over a Riemannian manifold M at a point p. Let Hol_p denote the associated holonomy group at p and let L_p be the group-norm given by (5.1). *Then (E_p, Hol_p, L_p) is a holonomic space.*

Holonomy space revisited

Definition

The function L_p, defined by (5.1) will be called *length norm* of the holonomy group induced by the Riemannian metric at p.

Theorem

Let E_p be the fiber of a vector bundle with metric and connection E over a Riemannian manifold M at a point p. Let Hol_p denote the associated holonomy group at p and let L_p be the group-norm given by (5.1). Then (E_p, Hol_p, L_p) is a holonomic space. More importantly, if E is endowed with the corresponding Sasaki-type metric, the associated holonomic distance coincides with the restricted metric on E_p from E.

2D Isoperimetry—with or without densities.

Consider a Riemannian surface (Σ, g) with volume element ω and suppose that there exists a rank-two vector bundle over Σ with connection such that the curvature 2-form is given by ω.

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Example

Consider a surface with metric g_0 and density μ. Let E be the tangent bundle together with the bundle metric $h = \mu g_0$ over the Riemannian metric $g = \mu^2 g_0$.

2D Isoperimetry—with or without densities.

Lemma

Let (M^2, g) be a 2-dimensional Riemannian manifold and let $\gamma : [0, \ell] \subseteq \mathbb{R} \to M$ be any curve parametrized by arc length. Let k_g be a signed geodesic curvature of γ with respect to an orientation of $\gamma^* TM$. Let $\theta(t)$ be the angle between $\dot{\gamma}$ and its parallel translate at time t. Then

$$2\pi - \theta(t) = \int_0^t k_g \tag{6.1}$$

Assume further that γ is a loop. Then, possibly up to a reversal in orientation, the holonomy action of γ at $p = \gamma(0)$ is the rotation by $2\pi - \int_0^\ell k_g$.

Thank you!