Infinite Graph Laplacians

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Dido Conference, May 26, 2010
Gammarth- Carthage

To the Memory of my Father
Prof. Dr. Ing. Bèchir Torki,
1931 – 2009

This work had been done basically at Fourier Institute *(Grenoble)*.

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I am grateful to Professor Yves Colin de Verdière who helped me to find my way in mathematic research and gave me the opportunity to have a joint work.

Introduction

Preliminaries

Infinite Graphs are considered in many domains not only in Mathematics :

1. Combinatorial and geometrical group theory,
2. Number theory,
3. General and algebraical topology,
4. Set theory,
5. Probability theory,
6. Mathematical physics

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1. Medical researchs (brain cells, bloody veins, retinal mosaicing..)
2. Chemistry,
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4. Electrical networks,
5. Synthetic imagery,
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7. Telecom,
8. Social sciences.

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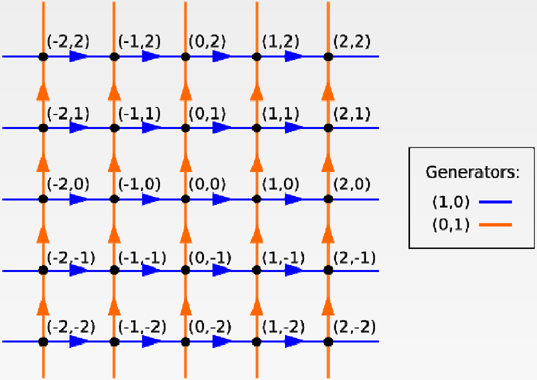
Let's see some examples :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Generators:
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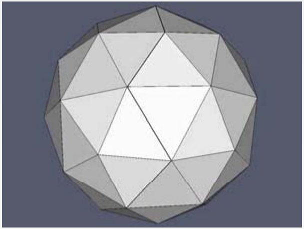
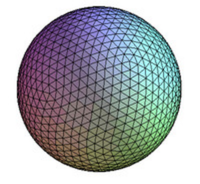
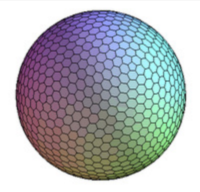
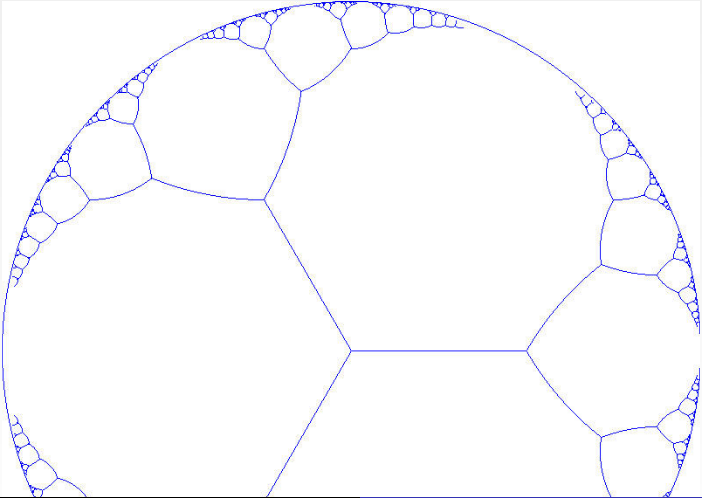


The subdivided icosahedron.
(80 triangles. Maximum edge length is 0.618.

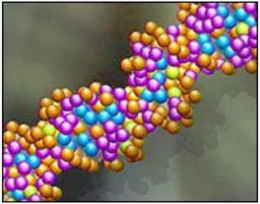
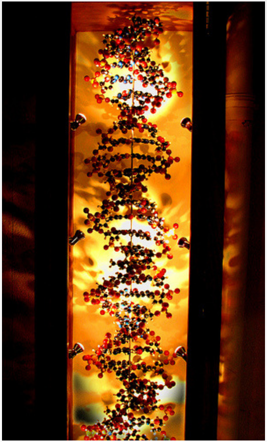
Graph as surface triangulation

triangulation		géode en nid d'abeille	

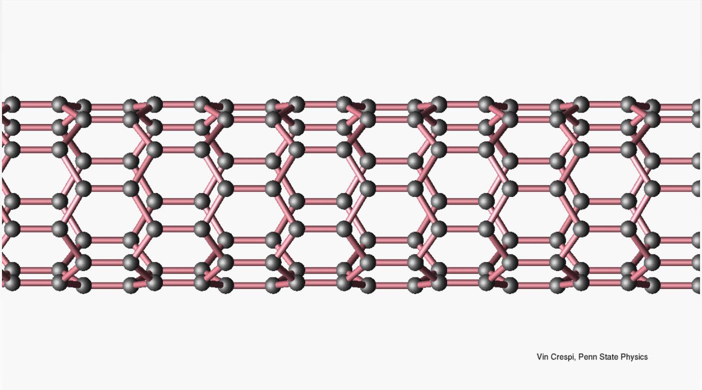
As Fractal



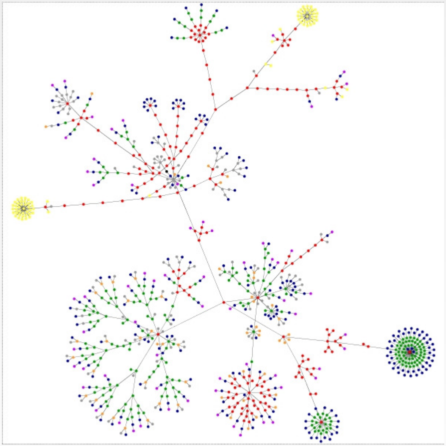
DNA graphs

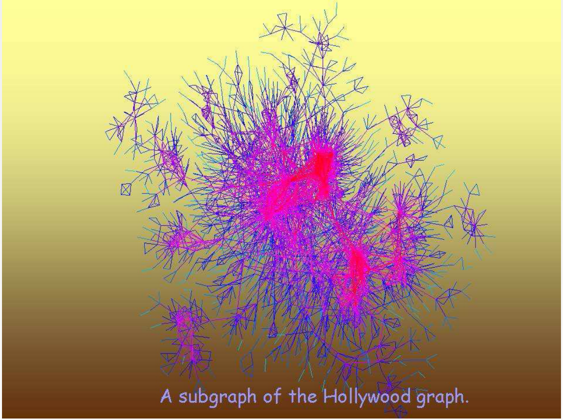
Carbon Nanotube



Website graph (Amazon's.com)

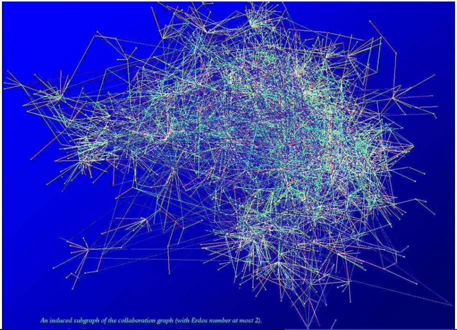


Hollywood Graph



Vertex: actors and actress
Edges: co-playing in the same movie
Only 10.000 out of 225.000 are shown.

Collaboration Graph



An induced subgraph of the collaboration graph (with Erdos number at most 2).

Visitors to math.arizona.edu/~dido

13 Sep 2009 to 13 May 2010:
2,874 visits shown above

Statistics updated 16 May 2010 at 09:24GMT: 2,945 visits
Total since 13 Sep 2009: 2,945

Motivation

This talk is a small contribution in Laplacians and Schrödinger operators on infinite graphs. Its the first of three works dealing with spectral theory of Laplacians and Schrödinger operators on infinite graphs.

We would like to find some analog results to those for non compact Riemannian manifolds.

There are many reasons to be interested in the essentially self-adjointness :

- If the operator is self-adjoint then its eigenvalues are real.
- Self-adjointness implies uniqueness of the solution of the Schrödinger operator in quantum mechanics.

One main proved result is an extension of Gaffney's theorem :

Theorem

The Laplacian of a complete Riemannian manifold is essentially self-adjoint.

Ref. : A special Stokes's theorem for complete Riemannian manifolds, Ann. of Math. **60** :140–145 (1954).

Remind that :

Definition

An unbounded symmetric linear operator on a Hilbert space is essentially self-adjoint **(ESA), if it has a unique self-adjoint extension.**

Among the results of essentially self-adjointness, we have :

1. Weyl (1909) in the case of $\mathbb{R}^n$.
2. Other classical results are cited in the famous four volume book of M. Reed and B. Simon.
3. M. Gaffney (1954) and R. Strichartz (1983) proved results for Laplacians of complete Riemannian manifolds .
4. T. Kato (1972)
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What's a Graph ?

What's a graph ?

A graph $G = (V, E)$ is a pair of two sets :

- V : the set of vertices.
- E : the set of edges which are pairs of vertices of V.
- $x \sim y$ indicates that the vertices x and y are joined, we call them adjacent or neighbors.
- The degree (or valency) of a vertex x is the number d_x of its neighbors.
- G is of bounded degree if there is an integer N such that the degree of each vertex is bounded by N.

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- If the graph G has a finite set of vertices , it's called a finite graph.
- Otherwise G is called infinite graph.

Convention

- G is connected : if given a partition (V_1, V_2) of the vertices into non-empty sets, there is an edge between $(V_1$ and $V_2)$.
- G is a simple graph : no multiple edge nor loop.
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Function spaces

- $C(V) = \{f : V \longrightarrow .\mathbb{R}\}$
- $C_0(V)$ is the subset of functions with finite support.
- For a weight function $\omega : V \longrightarrow \mathbb{R}_+^\star$,

$$l_\omega^2(V) = \left\{ f : V \longrightarrow \mathbb{R}; \sum_{x \in V} \omega_x^2 |f(x)|^2 < \infty \right\}.$$

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Graph Laplacian
Combinatorial Laplacian

Infinite Graph Laplacian

Definition

The Laplacien of a graph G weighted by $\omega : V \longrightarrow \mathbb{R}_+^\star$ and by $c : E \longrightarrow \mathbb{R}_+^\star$ is the operator $\Delta_{\omega,c}$ on $l^2_\omega(V)$ given by :

$$\left(\Delta_{\omega,c} f\right)(x) = \frac{1}{\omega_x^2} \sum_{\{x,y\}\in E} c_{\{x,y\}} \left(f(x) - f(y)\right)$$

It's a discrete version of the Laplace-Beltrami operator. They have strong connection to differential geometry.

We recall that on finite graph the Combinatorial Laplacian is defined by a symmetric matrix

$$A = (a_{i,j})$$

where $a_{i,j} < 0$ if $\{i, j\}$ is an edge and $a_{i,j} = 0$ if i and j are not neighbors.
Ref. : "Spectre de graphes", Y. Colin de Verdière (1998).

Main properties

- $\Delta_{\omega,c}$ is symmetric when considered with domain $C_0(V)$.
- Its quadratic form :

$$Q_c(f) = \sum_{\{x,y\}\in E} c_{\{x,y\}}(f(x) - f(y))^2$$

  is positive.
- $(\Delta_{\omega,c} f)(x)$ depends only on the neighbors of x.

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First result

Theorem

If the weight ω is constant then the operator $\Delta_{\omega,c}$ with domain $C_0\left(V\right)$ is essentially self-adjoint.

We should use the following lemma :

Lemme

The positif symmetric operator $\Delta : C_0\left(V\right) \to l^2(V)$ is essentially self-adjoint iff $Ker(\Delta^\star + 1) = \{0\}$

$$Dom(\Delta^\star) = \{f \in l^2(V); \Delta f \in l^2(V)\}$$

The proof of the theorem is based on an idea taken in the PhD thesis of R.K. Wojciechowski :"Stochastic completeness of graphs" (2008).

proof

- $\Delta^{\star} f(x) = \Delta f(x)$
- Let g a function on V : $\Delta_{\omega_0,c} g + g = 0$
- Suppose that x_0 is such that $g(x_0) > 0$.
- $\Delta_{\omega_0,c} g(x_0) + g(x_0) = 0$.
- $\frac{1}{\omega_0^2} \sum_{y \sim x_0} c_{x_0,y}(g(x_0) - g(y)) + g(x_0) = 0$.
- There's a vertex x_1 , $g(x_0) < g(x_1)$.
- Same thing to x_1.
- We obtain by induction a sequence $(g(x_n))_n$ positive strictly increasing .
- The function g can not be in $l^2(V)$.
- We can do the same if $g(x_0) < 0$.

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Combinatorial Schrödinger operator

What's a Schrödinger operator on a graph ?

Definition

A Schrödinger operator on G is an operator of the form

$$H = \Delta_{1,a} + W$$

where W is a real function on V.

Theorem

Every Schrödinger operator on $l^2(V)$ which is positive definite as a quadratic form in $C_0(V)$ is unitarily equivalent to a Laplacian $\Delta_{1,a}$ on the graph.

The proof of this Theorem uses the following Lemma :

Lemme

If H is a Schrödinger operator on $l^2(V)$ which is positive definite as a quadratic form in $C_0(V)$ then there exists a $H-$harmonic strictly positive function on V.

The proof of the Lemma requires some graph results analog to those known in manifolds :

- Harnack inequality.
- Minimum principle.
- Dirichlet problem.

Essential self-adjointness

We first define a special distance on the graph :

Definition

Let a a strictly positive function on V. We define a-weighted-distance on G the distance δ_a, given by :

$$\delta_a(x,y) = \min_{\gamma \in \Gamma_{x,y}} L(\gamma)$$

where $\Gamma_{x,y}$ is the set of every way $\gamma : x_1 = x, x_2, \ldots x_n = y$ relying x to y ; and $L(\gamma) = \sum_{1 \leq i \leq n} \frac{1}{\sqrt{a_{x_i x_{i+1}}}}$ is the length of γ.

One of the main results for complete metrically graphs is the following Theorem :

Theorem

Let $H = \Delta_{1,a} + W$ a Schrödinger operator on an infinite graph G with bounded degree such that its metric defined by the distance δ_a is complete. We assume that there is a reel constant k satisfying $\langle Hg, g\rangle \geq k\|g\|^2_{l^2}$, for any $g \in C_0(V)$. Then the operator H, with domain $C_0(V)$, is essentially self-adjoint.

Using the fact that a Schrödinger operator is unitarally equivalent to a Laplacian, we have :

Theorem

Let G an infinite graph with bounded degree and $\Delta_{\omega,c}$ a Laplacian on G. We assume that the metric given by the distance δ_a is complete, where a is the function defined by

$$a_{\{x,y\}} = \frac{c_{\{x,y\}}}{\omega_x \omega_y}.$$

Then the operator $\Delta_{\omega,c}$, with domaine $C_0(V)$, is essentially self-adjoint.

Continuation

In a joint work with Yves Colin de Verdière and recently with Françoise Truc, we have some other essentially self-adjointness results of :

- Schrödinger operator on noncomplete graphs.
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Questions ?....