GEOMETRIC INTERPRETATION AND UTILITIES RELATED TO THE MONGE-AMPERE EQUATION

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OUTLINES

- Equations of Monge-Ampère type
- Geometric interpretations, the Weyl and Minkowski problem
- History and utilities
- Results on existence and regularity of solutions
    - in bounded domains
    - in unbounded domains

Equations of Monge-Ampère type

- A real Monge-Ampère equation is a fully nonlinear second order partial differential equation of special kind:

$$\det D^2 u = f(x,u,Du)$$

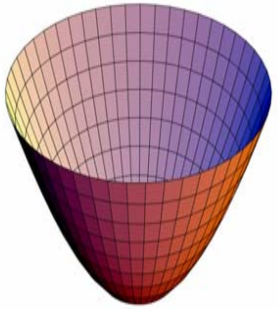
Ampère

x: a variable with values in IR^n

u: a function defined over a domain Ω of IR^n

► When **f≡1**, an entire convex solution must be a quadratic polynomial

(by Calabi for n≤5 and Pogorolov for n)



Geometric interpretations, the Weyl and Minkowski problem

- Monge-Ampère equations arises naturally in several problems in Riemannian geometry, conformal geometry, and CR geometry.
- The simplest application is the problem of prescribed Gauss curvature:

Suppose a real valued function K is specified on a domain Ω in IR^n, we seek to identify a hypersurface of IR^{n+1} as a graph: **z=u(x)** over $x \in \Omega$ so that:

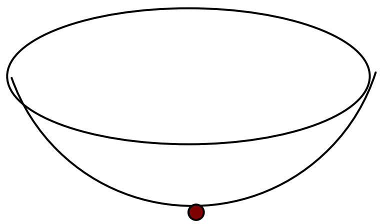
$$\det D^2u = K(x)(1+|Du|^2)^{(n+2)/2}$$

- If $z=u(x,y)$ is the equation of a hypersurface $\sum$ of IR^3,

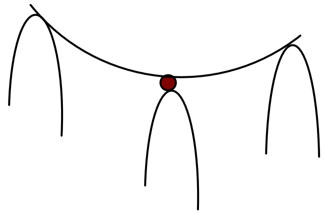
$$\det D^2u = u_{xx}u_{yy}-u_{xy} = k_1(x,y)k_2(x,y)$$

where k_1, k_2, are the eigenvalues of D^2u, the point (x,y) is

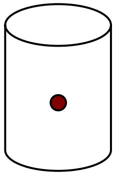
- **elliptic** when $k_1k_2>0$, in that case u is convex near (x,y)



- **hyperbolic** when $k_1k_2<0$



- **parabolic** when $D^2u\neq0$ and $k_1k_2=0$



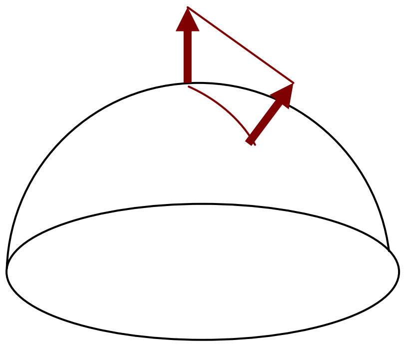
- Denote by **N** the unite normal to the surface:

Σ: **z=u(x,y),**

$$N=1/(1+|Du|^2)\ (-Du,1)$$

- Principle curvatuves $k_1(x,y)$, $k_2(x,y)$ are the eigenvalues of the Weirgarten map (shape form map),

$$W: T_{(x,y)}\Sigma \to T_{(x,y)}\Sigma$$
$$v \mapsto N'(x,y)v$$

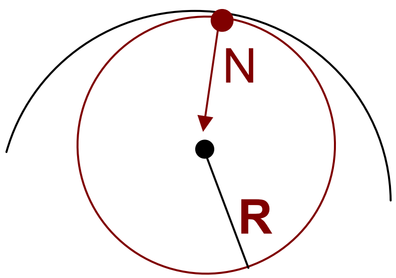


- The **Gauss curvature** is defined by:

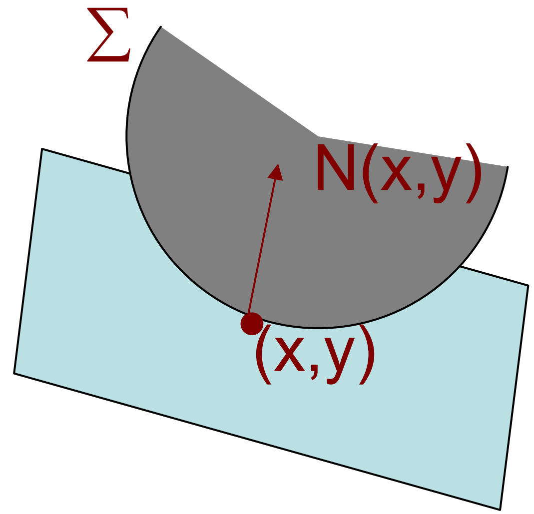
$$K = k_1 k_2 = \det(D^2u)/(1+|Du|^2)^{(n+2)/2}$$

- When **K>0**, (x,y) is **elliptic** and if $k_1>0$, $k_2>0$, the surface is **convex** near (x,y)
- When $k_1 = k_2$, (x,y) is called **spherical** point

Osculator circle to a curve at a point having principle curvature k>0



R=1/k

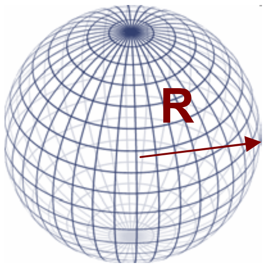


An elliptic point

- **The Minkowski problem (1903) goes as follows:**

Given f defined on S^2, find a strictly convex surface: $\sum \subset IR^3$ such that the Gauss curvature of $\sum$ at the point x equals f(N(x))

- When f(x) is identically equal to the constant: $1/R^2$, the solution would be the sphere of radius R



➢ The Weyl problem (1916) goes as follows:

Given a Riemannien metric g on S^2 having Gauss curvature $K>0$, we seek a regular isometric embedding wich image is a convex hypersurface Σ :

$$X:(S^2,g)\rightarrow(IR^3,h)$$

h: the standard plane metric

History and utilities

- **For the Minkowski problem: (1903)**

- Minkowski prouved the uniqueness and tried to prove existence of such surface as a limit of polyhedrons

- 1938, Lewy, under hypothesis of analicity of K, proved the existence and uniqueness of the required surface

- 1971, Pogorelov studied the generalized Minkowski problem consisting in finding a convex closed hypersurface in IR^n with prescribed Gauss curvature K

➢ For the Weyl problem (1916):

- ❖ Weyl tried to solve it by classical method of continuity based on à priori estimations of holderian norms of smooth solutions
- ❖ 1962, Niremberg completed this proof and solve the existance problem
- ❖ 1948, Alexandroff obtained a generalized solution to Weyl problem as a limit of polyhedrons.
- ❖ 1949, Pogorelov proved the regularity of Alexandroff solution
- ❖ 1984, Oliker, generalized Weyl problem to dimention n:

$$\mathbf{Det}\ D^2u = K(x)(1+|\nabla u|^2)^{(n+2)/2}$$

Utilities in Physics

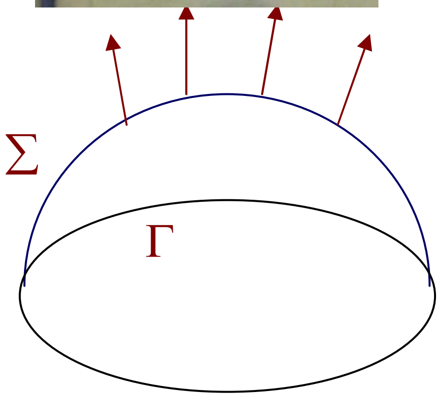
a belgian physist.

- 1850, Joseph Antoine Ferdinand Plateu,

Plateau's Problem: Determine the shape of the minimal surface constrained by a given boundary, there arizes then the question:

Is there a convex surface with prescribed Gauss curvature a constant K>0 and its boundary is a Jordan curve?

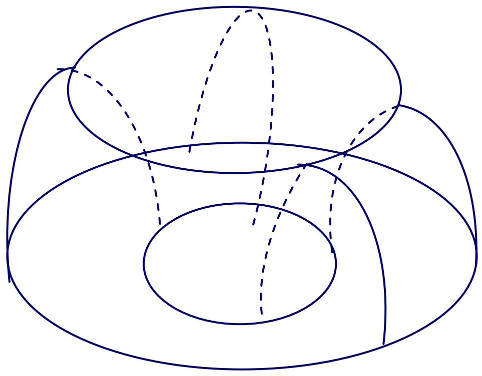
- Caffarelli, Nirembeg, Spruck in 1984, prouved existance an unicity of such surface



The surface Σ of the sop film is submitted to the homogenous presser of the air

- 1992, Hoffman, Rosenberg, Spruck, generalized that result for graphs above rings planes (Σ is not global convex)

- 2004, Guan and Spruck, proved that two prallel planes limit a K-surface which topology is of the ring



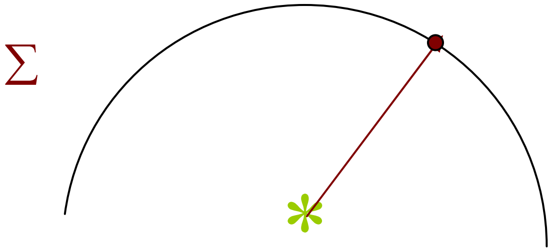
- In geometrical optic it rises these two following related problems:

- Given the light intensity I of a light ray R, find the wave surface Σ:

the wave surface Σ is orthogonal to the light beam and the light intensity at point A is proportional to the Gauss curvature of Σ at point A :

$$I=cK$$

Can burn for bad Gauss curvature of surface Σ



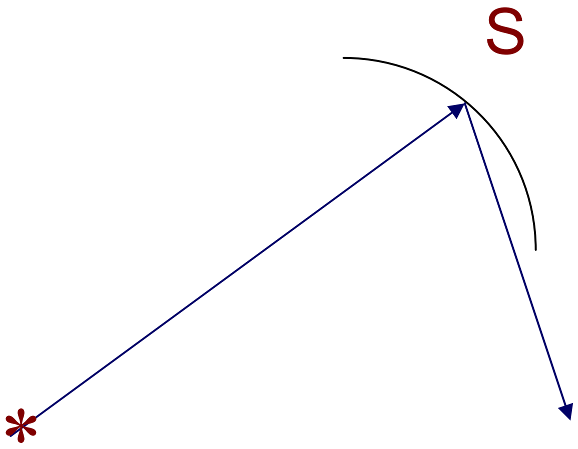
- Suppose a homogenous light source is located at the origin, $I(m)$ the intensity of this light source in the direction m and suppose that a ray reaches the surface Σ at a point $r(m)$ and is reflected in the direction $y(m)$. The law of reflection is:

$$y(m)=m-2(m|N)N$$

Which defines a function y from S^1 into IR^3. The intesity of the reflected light is given by:

$$I(y(m)) = I(m)/\det(D^2y)$$

- **Thus arises that question: how do we reconstruct a surface Σ from the light rays reflected? (Oliker)**



principle of the device antenna reflector considered by some engineers is also linked to this type of equation

Results on existence and regularity of solutions

➢ **Case when Ω is a bounded domain :**

We consider then the Dirichlet problem:

$$\begin{cases} \det D^2 u = f(x,u,Du) > 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

- 2008, N.S. Trudinger, X.J.Wang,

if **$f=f(x)$,** Ω is a convex domain in IR^n, $\partial\Omega \in C^3$, $\phi \in C^3(\overline{\Omega})$, $\inf f > 0$ and $f \in C^\alpha(\overline{\Omega})$ then there exists a convex solution u:

$$|u|_{2,\alpha} \leq C$$

- 1999, P. Guan, N.S. Trudinger, X.J.Wang: if $f=f(x) \geq 0$, $\partial\Omega$, $\phi \in C^{3,1}$ and $f^{1/n-1} \in C^{1,1}$, then u exists

- when **$f=f(x,u,\nabla u)>0$,**

- 1984, L.Caffarelli, L.Niremberg, J.Spruk,

If there exists a subsolution

$$\underline{u}: \det D^2 \underline{u} \geq 0 \quad \text{in } \Omega, \qquad \underline{u}=\phi \quad \text{on } \partial\Omega$$

and Ω is a convex domain, there exists a unique solution u:

$$|u|_{2,\alpha} \leq C$$

- 1998, B.Guan, generalized the latter work to **non convex domains**, nevertheless had no result of **existance of subsolutions**

- 2004, B.Guan, obtained some good results for infinit boundary value even in unbounded domains

➢ Case where the domain is unbounded:

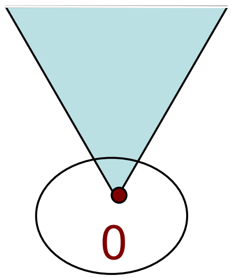
❖ 1996, K.S.Chou, X.J.wang,

when $f=f(x)$ and $0<c_1\leq f\leq c_2$, there exist infinitely many entire convexe solutions to the MA equations in IR^n

- Existance of entire convex solutions for any given positive function f is still open problem

❖ 2002, F.Finster, O.C.Shurer,

If there exists a subsolution $\underline{u}$ close to a cone, there exists a hypersurface of prescribed Gauss curvature in an exterior domain which is close to a cone: $|x| \leq u \leq |x|+\varphi$



$z=|x|$

Has a derivative not bounded near 0

- **2010, Lamia Bel Kenani Toukabri,** J. Math. Anal. Appl. 363 (2010) 596–605, jointed work with Saoussen Kallel-Jallouli published the following result of existence of entire solution for **n**≥**3**,

if **f** *>0* in $C^2(IR^n \times IR \times IR^n)$. We prove the existence of convex solutions, provided there exists a subsolution of the form $\underline{u} = a|x|^2$ and a superharmonic bounded positive function φ satisfying:

$$f \geq (2a + \Delta \varphi / n)^n.$$

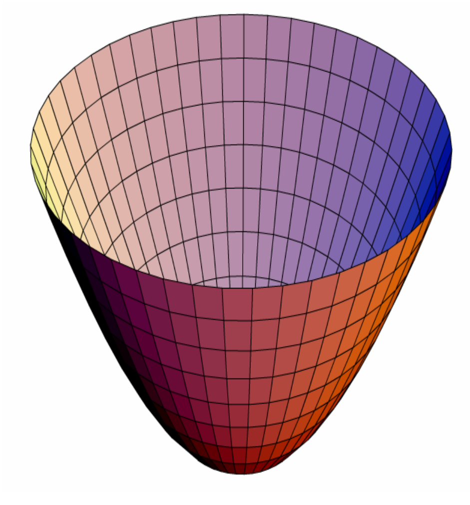
- The hypothesis on **f** is so that

$$a|x|^2 \leq u \leq a|x|^2 + \varphi$$

and D^2u is uniformly bounded,

with φ superharmonic bounded positive function

then the graph of the surface Σ is close to a paraboloïd



THANK YOU