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Hardy Type Inequalities for a Special Family of Non-convex Domains.

M.Hoffmann-Ostenhof, T.Hoffmann-Ostenhof, A. Laptev,

"A Geometrical Version of Hardy's Inequality", 2001:

Theorem 1. For any domain $\Omega \subset \mathbb{R}^n$ and any function $u \in H_0^1(\Omega)$ we have the following inequality:

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{n}{4} \int_\Omega \int_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^2(x)} |u(x)|^2 dx + \frac{K(n)}{|\Omega|^{2/n}} \int_\Omega |u(x)|^2 dx \ , \tag{1}$$

$\mathbb{S}^{n-1}$ is the unit spere in $\mathbb{R}^n$,

$d\omega(\nu) : \quad \int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1, \quad \nu \in \mathbb{S}^{n-1}$,

$\tau_\nu(x) = \min\{s > 0 : x + s\nu \notin \Omega\}$,

$\rho_\nu(x) = \min\{\tau_\nu(x), \tau_{-\nu}(x)\}$,

$|\Omega| := volume(\Omega)$,

$$K(n) = \frac{n^{(n-2)/n} \ |\mathbb{S}^{n-1}|^{2/n}}{4},$$

$|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere.

Theorem 2. For any convex domain $\Omega \subset \mathbb{R}^n$ and for any function $u \in H_0^1(\Omega)$ the following inequality holds:

$$\int\limits_{\Omega} |\nabla u(x)|^2 dx \geq \frac{1}{4} \int\limits_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx + \frac{K(n)}{|\Omega|^{2/n}} \int\limits_{\Omega} |u(x)|^2 dx, \quad (2)$$

$\delta(x) := dist(x, \partial\Omega)$.

$\Omega \subset \mathbb{R}^n$ is called reqular domain, if

$$\exists\, c: \quad \delta(x) \le m(x) \le c\delta(x) \quad \forall x \in \Omega,$$

$$\frac{1}{m^2(x)} := \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\tau_\nu(x)}.$$

For regular domain $\Omega \subset \mathbb{R}^n$

$$\int\limits_{\Omega} |\nabla u(x)|^2 dx \ge \frac{n}{2c^2} \int\limits_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx + \frac{K(n)}{|\Omega|^{2/n}} \int\limits_{\Omega} |u(x)|^2 dx \quad (3)$$

$n = 2$

E.B.Davies "Spectral Theory and Differential Operators " (1995)

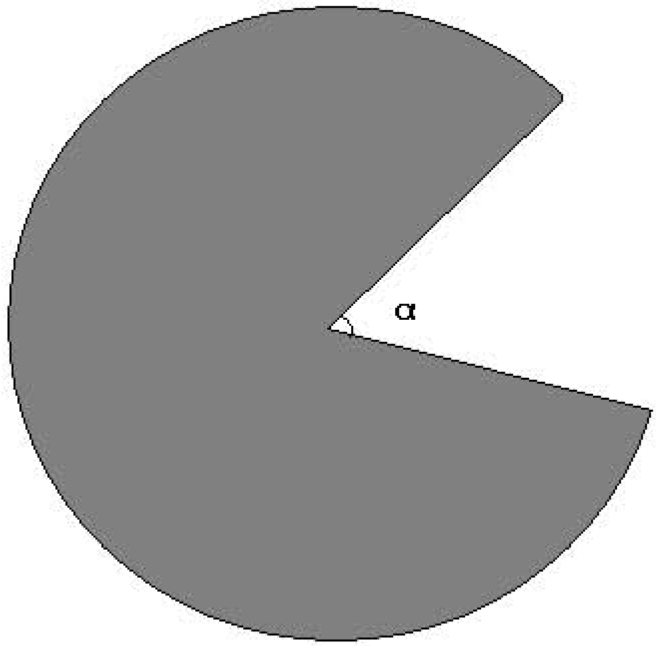
Theorem 3. The domain $\Omega \subset \mathbb{R}^n$ is regular, if there exists the constant $\widetilde{c} > 0$, such that

$$| \, \widetilde{\Omega}(a)| := volume\{y \notin \Omega : dist(y, a) < r\} \geq \widetilde{c} \, r^n,$$

$$\forall \, a \in \partial\Omega \quad \forall \, r : 0 < r \leq \delta_0(\Omega),$$

$$\delta_0(\Omega) := \sup_{x \in \Omega} \delta(x).$$

$$c = 2 \Big[\frac{|\mathbb{S}^{n-1}| \, (2^n - 1)}{\widetilde{c} \, n} \Big]^{1/2}$$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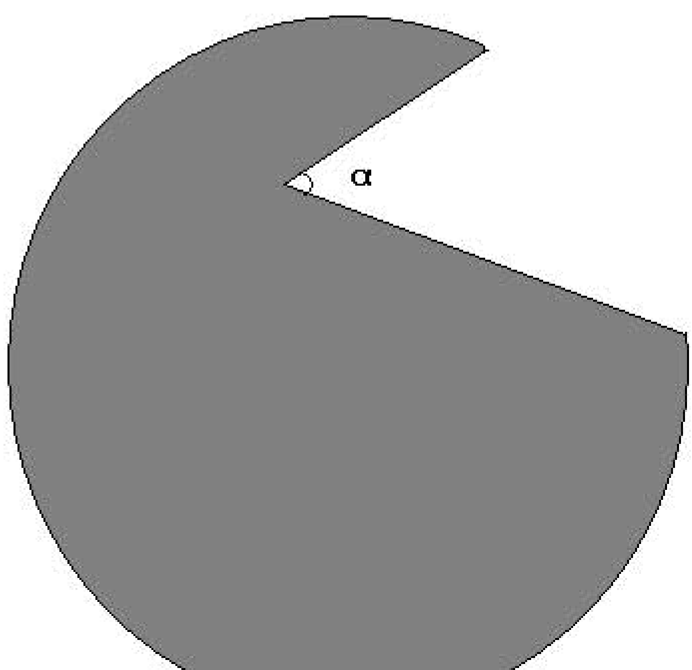
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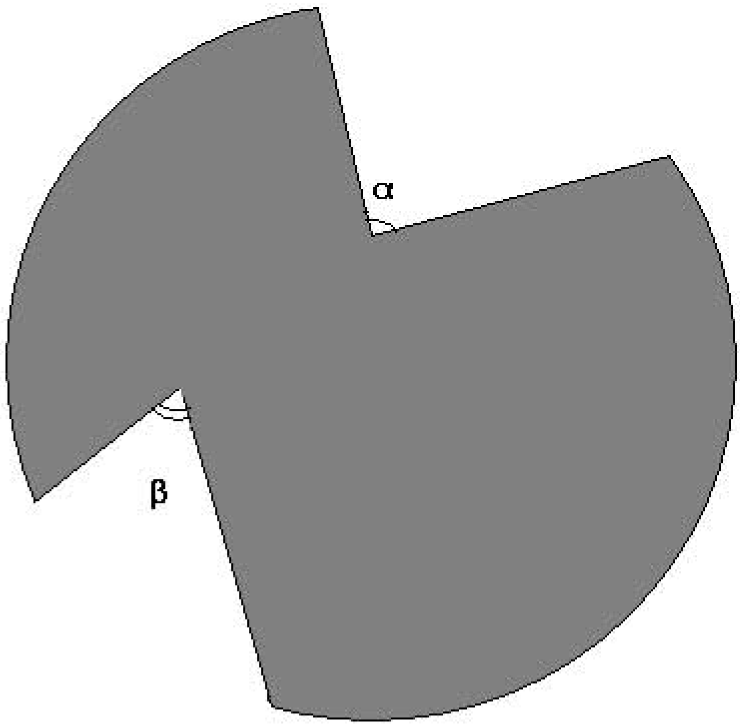
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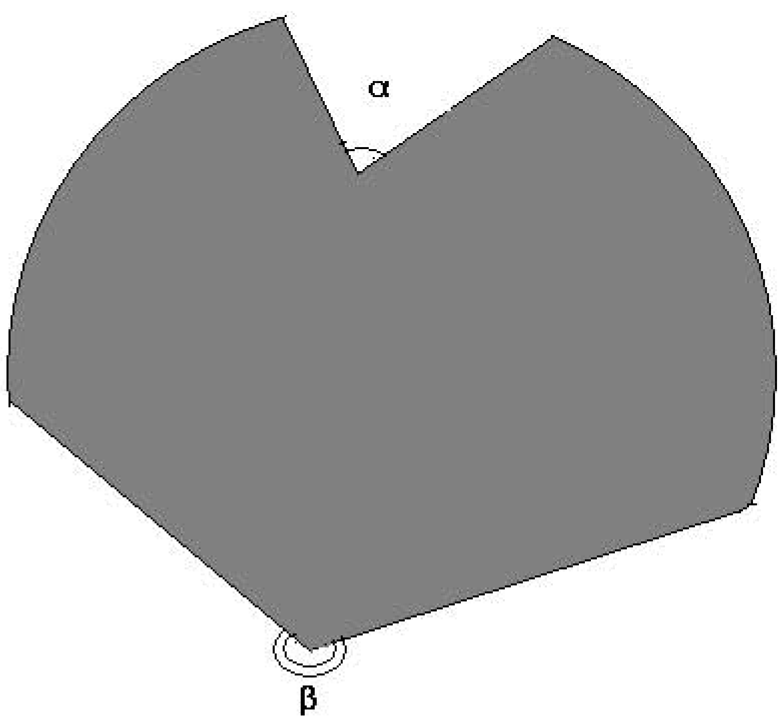
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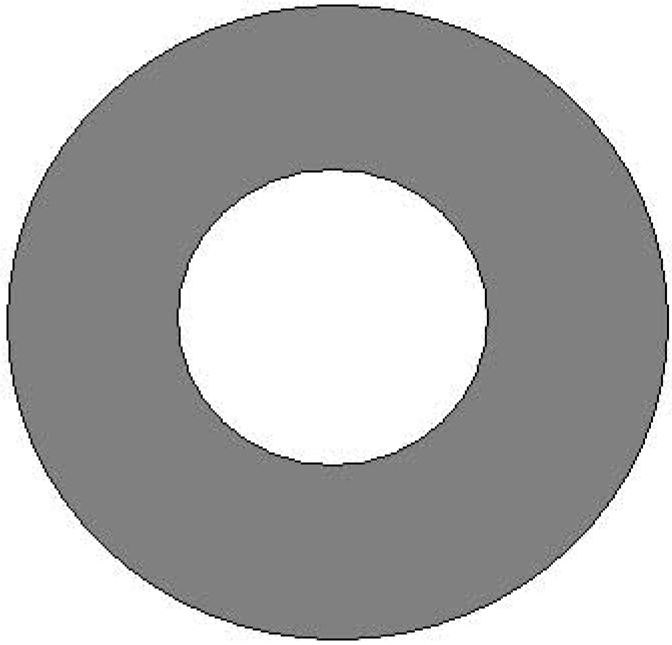

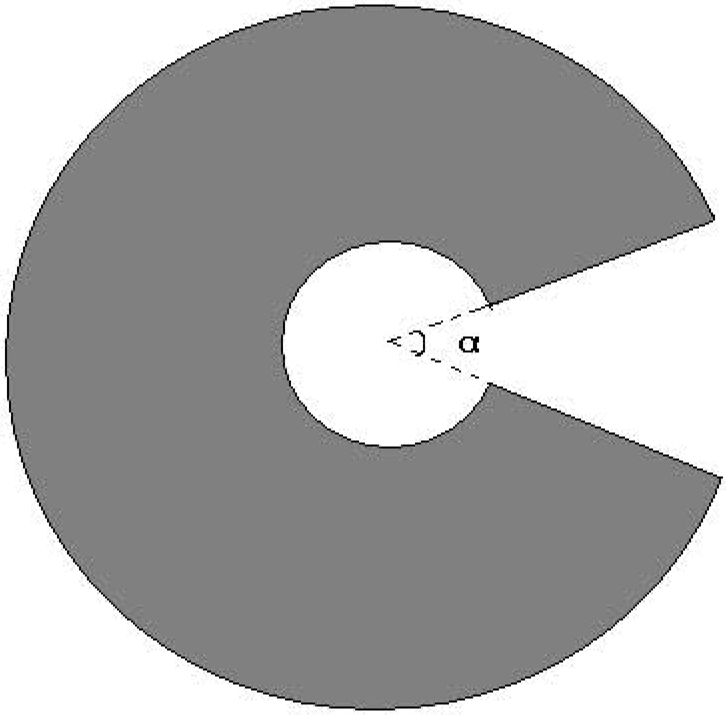
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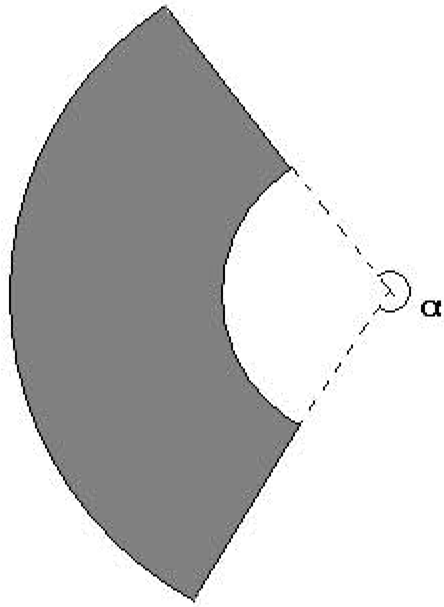
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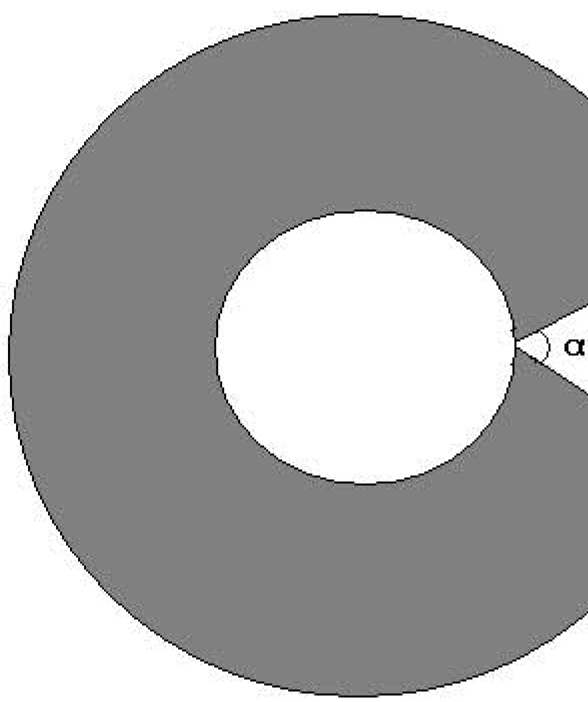
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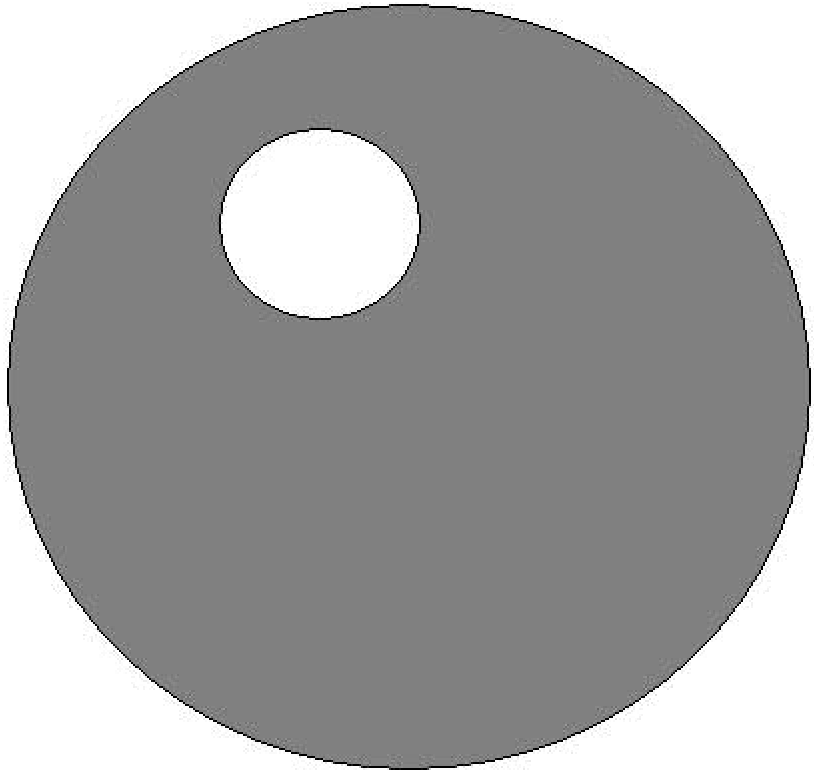

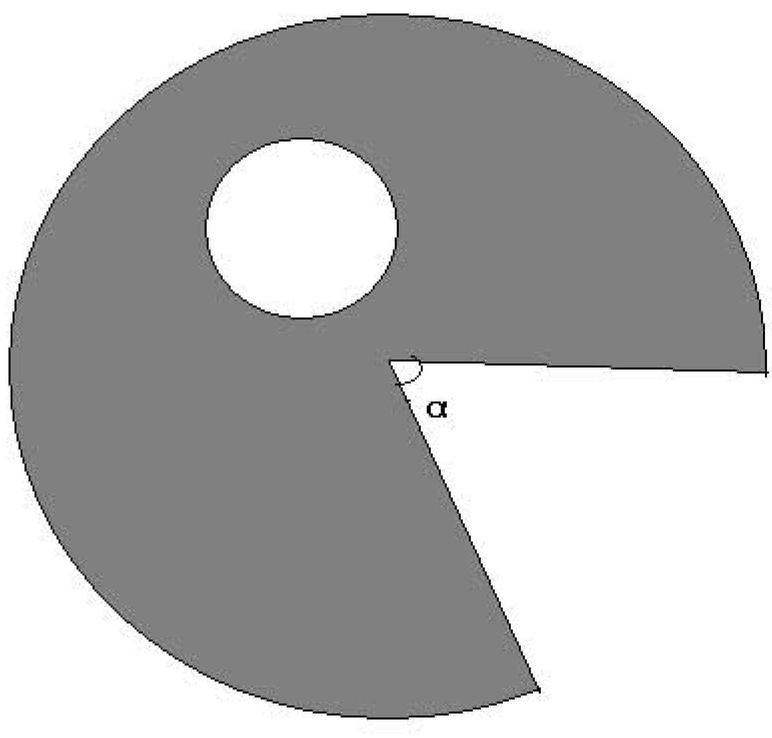
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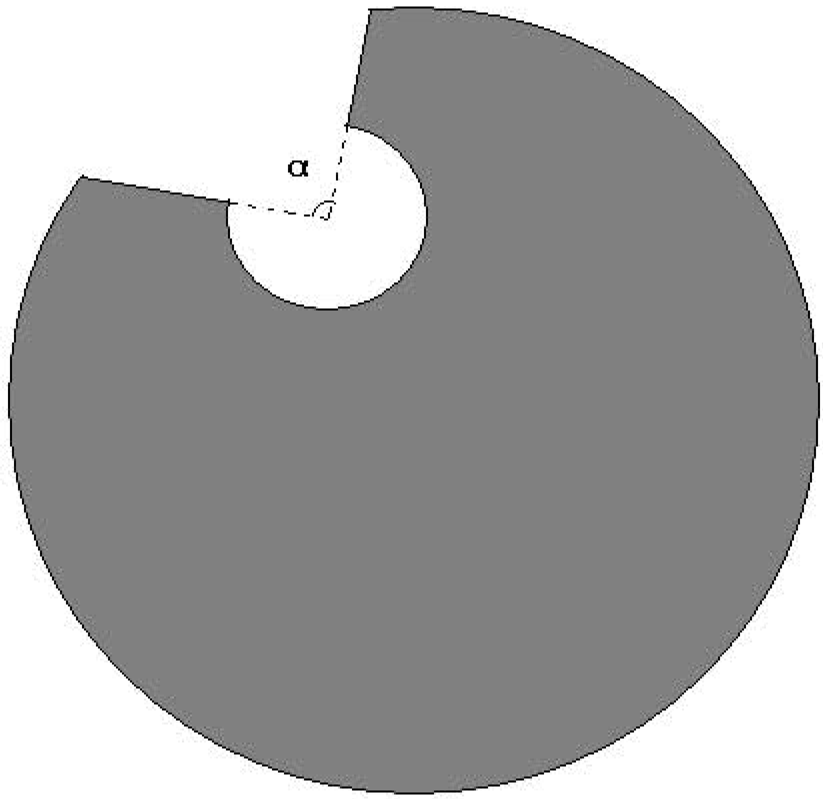
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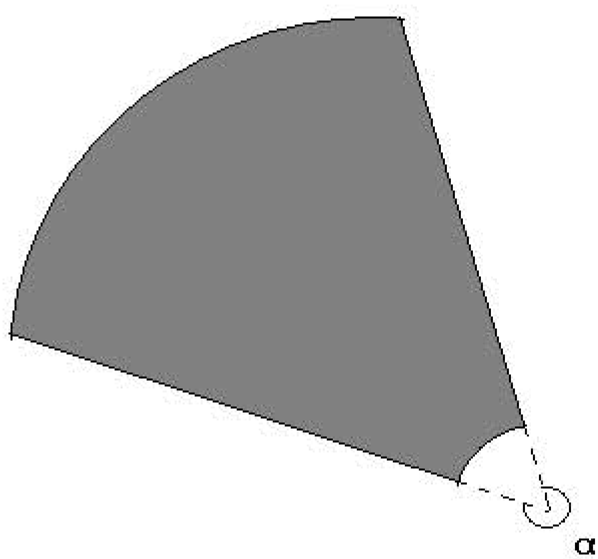
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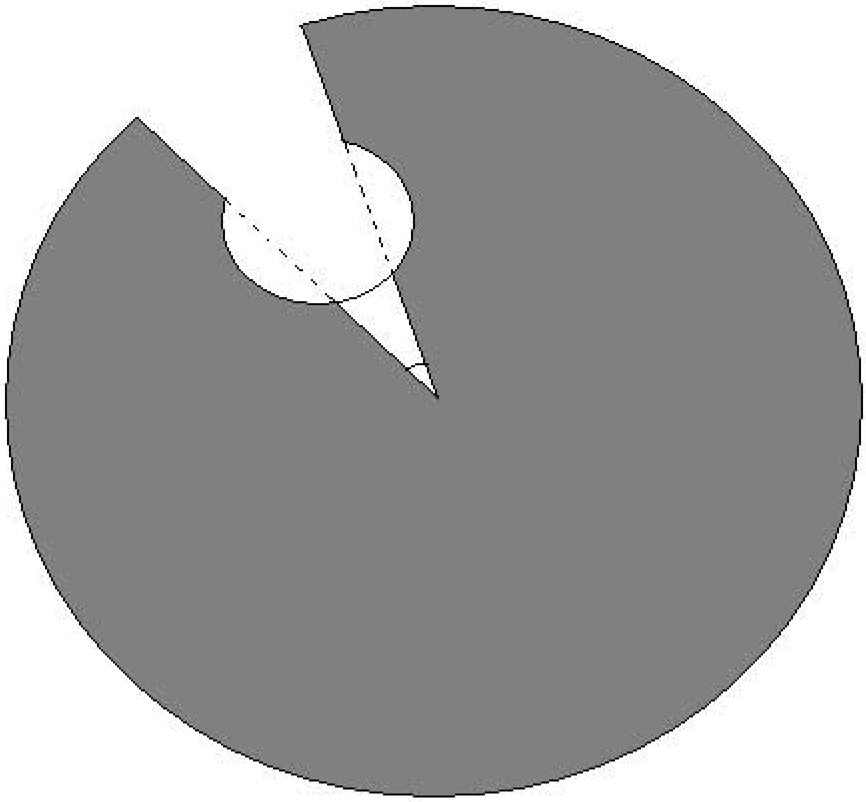

Ω is concentric annulus with radii R_1 и R_2 $(R_2 < R_1)$

$$R_2 \geq \frac{R_1}{5}, \qquad \widetilde{c} = \frac{\pi}{8},$$

$$\int\limits_{\Omega} |\nabla u(x)|^2 dx \geq \frac{1}{96} \int\limits_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx + \frac{\pi}{2|\Omega|} \int\limits_{\Omega} |u(x)|^2 dx.$$

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