

Trigonometry Lecture Notes

Section 5.1

Angles and Their Measure

Definitions:

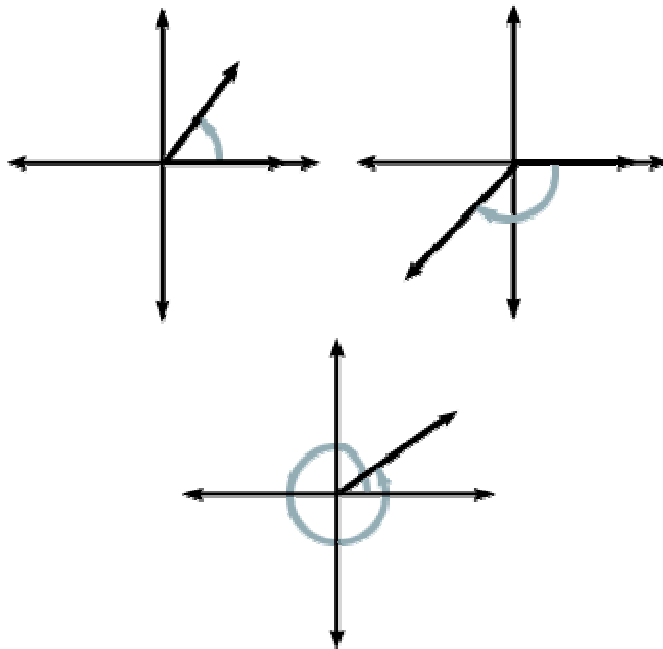
A **Ray** is part of a line that has only one end point and extends forever in the opposite direction.

An **Angle** is formed by two rays that have a common endpoint. One ray is called the **initial side** and the other is called the **terminal side**.

The common endpoint of an angle's initial side and terminal side is the **vertex** of the angle.

Standard Position: An angle is in Standard Position if

- Its vertex is at the origin of a rectangular coordinate system and
- Its initial side lies along the positive x-axis



Positive angles are generated by counter clockwise rotation, while **negative angles** are generated by clockwise rotation.

The angles illustrated above lie in Quadrants I (first and last drawing), and III (second drawing). Note not all angles lie in a quadrant.

Measuring Angles Using Degrees

If the minute hand of a clock goes from the 12 noon position all the way to the 12 midnight position it has traveled 360 degrees, denoted as 360° .



Special names for angles of certain measure:

An **Acute** angle measures less than 90 degrees.

A **Right** angle measures exactly 90 degrees.

An **Obtuse** angle measures more than 90, but less than 180 degrees.

A **Straight** angle measures exactly 180 degrees.

Example 1

Draw each of the following angles in standard position:

a. 45° b. 225° c. 90° d. 270° e. -135° f. 405°

We refer to angles with the same initial and terminal sides, like examples a. and f., as **coterminal angles**.

An angle of X° is coterminal with angles of $X^\circ + k \cdot 360^\circ$ where k is an integer.

Example 2

Find an angle less than 360 degrees that is coterminal with a. 430° b. -120°

Two positive angles are **Complements** if their sum is 90 degrees, and two positive angles are **Supplements** if their sum is 180 degrees.

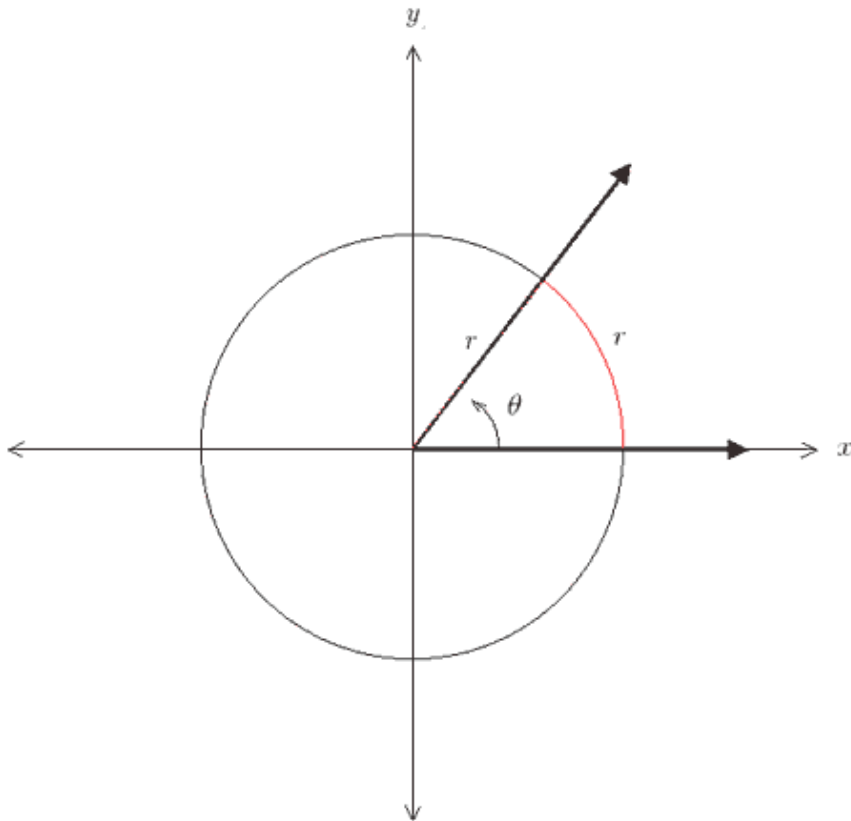
Example 3

Find the complement and supplement of 62°

Radians

Radians are another way to measure angles.

One **Radian** is the measure of the central angle of a circle that intercepts an arc equal in length to the radius of the circle. See the picture below:



The radian measure of any central angle is the length of the intercepted arc divided by the radius of the circle.

$$\theta = \frac{\text{length of the intercepted arc}}{\text{radius}} = \frac{s}{r} \text{ radians}$$

Example 4

A central angle, θ , in a circle of radius 6 inches intercepts an arc of length 15 inches. What is the radian measure of θ ?

**Note that the units cancel out!

Converting between Degrees and Radians

What is the arc length intercepted by a central angle that makes one complete rotation (a 360° angle)—i.e. its initial side and terminal side are in the same location, the positive x axis?

Well in this special case the arc length is the full circumference of the circle. Do you remember that basic formula from geometry? $C = 2\pi r$

Then the angle has measure 2π radians, why?

This leads us to the idea that $360^\circ = 2\pi$

We can then say that $180^\circ = \pi$, and finally we can use a convenient form of one, $1 = \frac{180^\circ}{\pi}$

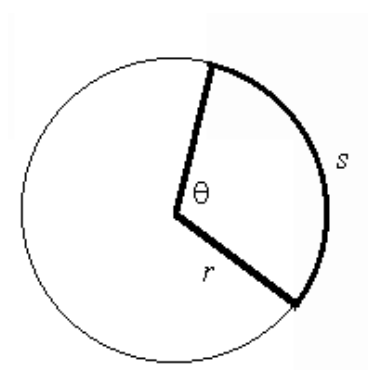
Example 5

Convert the following into radians: a) 60° b) -135°

Example 6

Convert the following into degrees: a) $\frac{\pi}{3}$ radians b) $\frac{-5\pi}{3}$ radian c) 1 radian

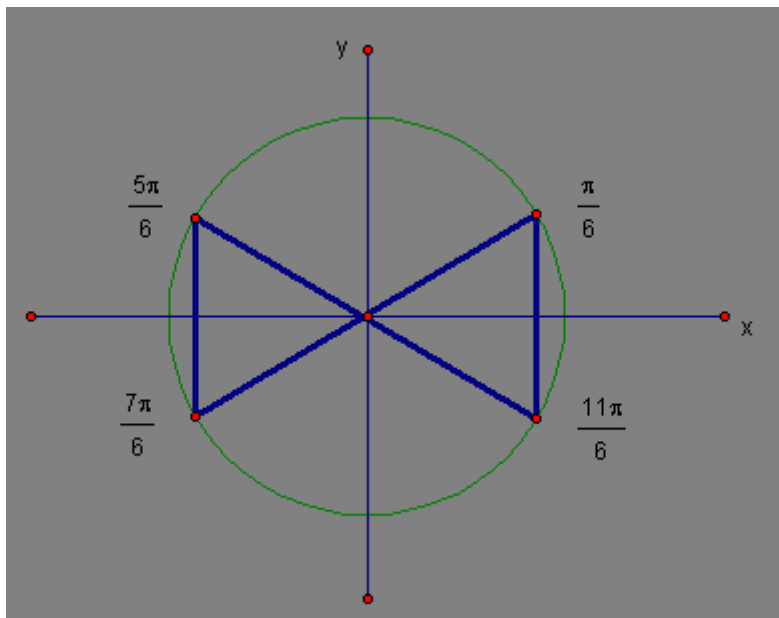
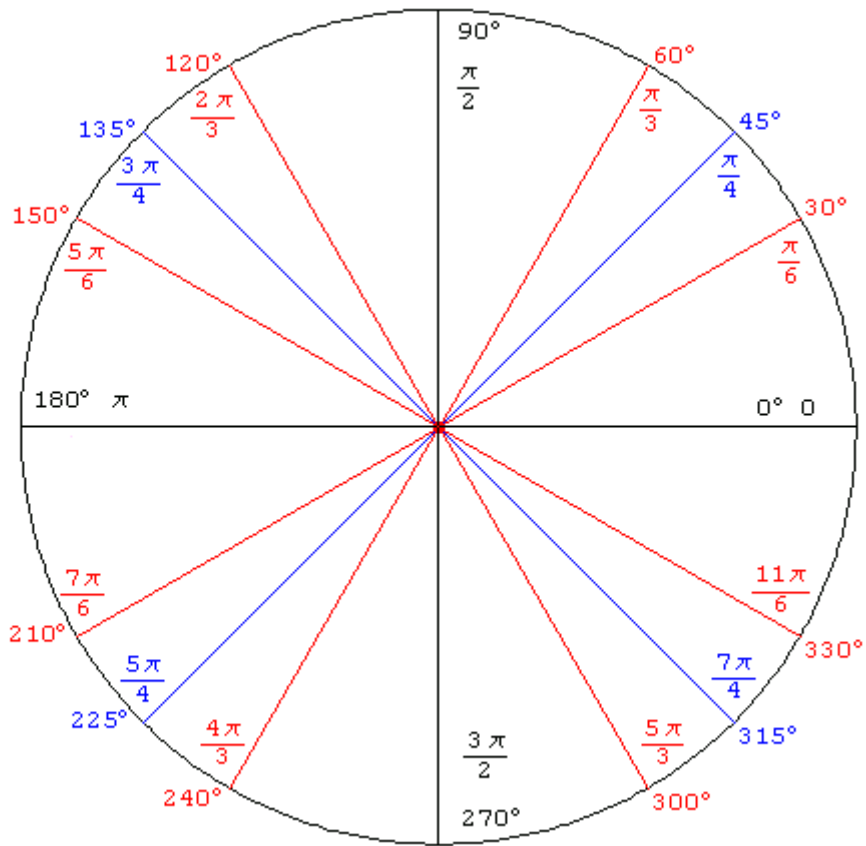
The Length of a Circular Arc

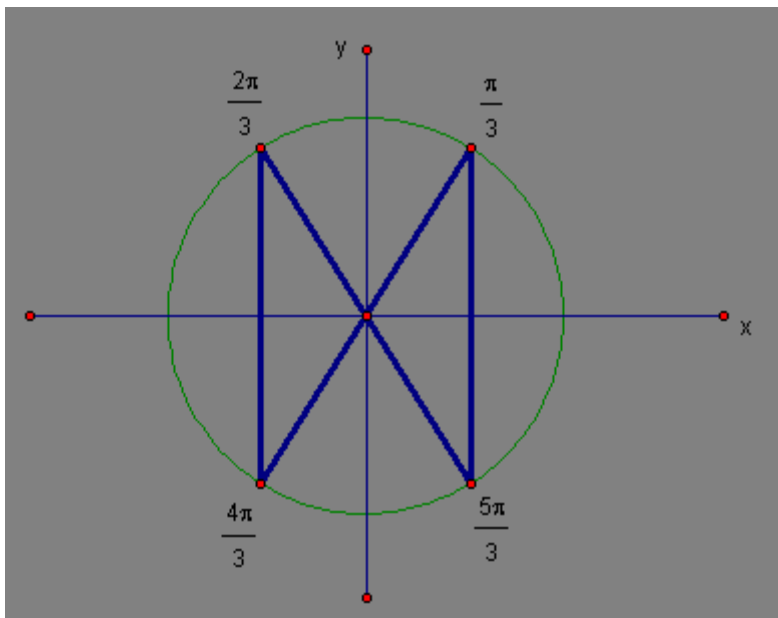
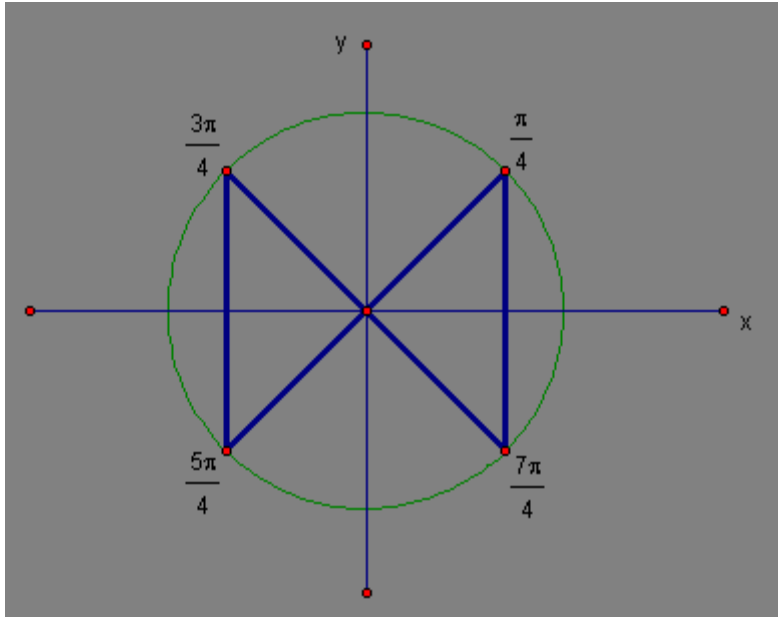


Recall that we said earlier that $\theta = \frac{\text{length of the intercepted arc}}{\text{radius}} = \frac{s}{r}$, well this leads to the following

formula: $s = r\theta$

We should memorize these diagrams



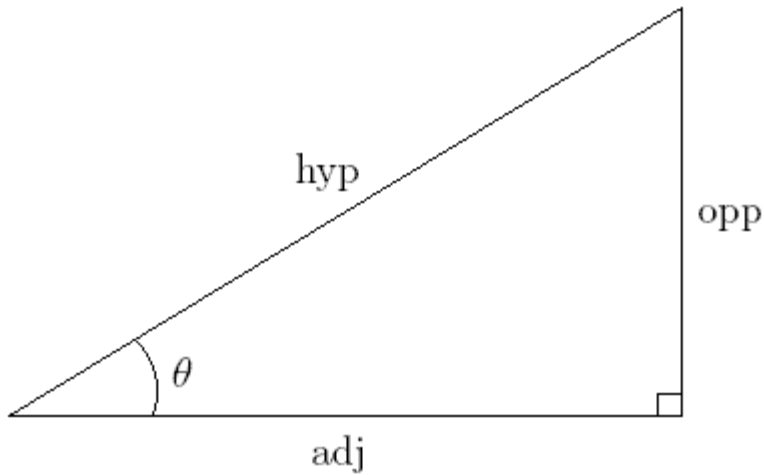


Example 6.5 Convert $63^{\circ}32'47''$ into decimal form. Then convert 58.325° into DMS form.

Section 5.2

Right Angle Trigonometry

Triangles have three angles. Some special triangles have a 90 degree angle (just one of course, since the sum of all angles in a triangle must add to 180 degrees). We want to be able to talk about some special names for the sides of these triangles relative to one of the other (non- 90°) angles. Consider the diagram below:

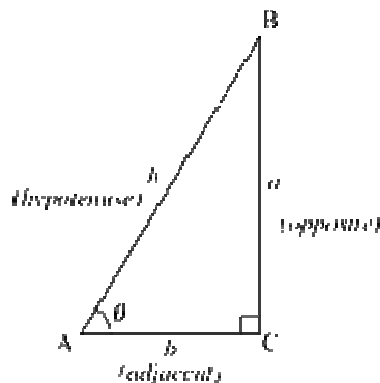


Now we can talk about the six trigonometric functions. These functions are like the functions you know of already, such as $f(X)$, but the six trig functions have names instead of the single letter names of f, g, and h.

Here are the names and their abbreviations:

Name	Abbreviation	Name	Abbreviation
Sine	Sin	Cosecant	Csc
Cosine	Cos	Secant	Sec
Tangent	Tan	Cotangent	Cot

Here is how we will define these functions:



$$\text{sine } \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{cosecant } \theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\text{cosine } \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{secant } \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\text{tangent } \theta = \frac{\text{opposite}}{\text{adjacent}}$$

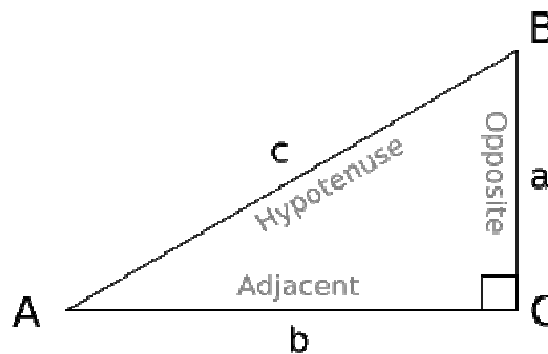
$$\text{cotangent } \theta = \frac{\text{adjacent}}{\text{opposite}}$$

Notice the functions in the left column (sine, cosine, and tangent) are reciprocals of the right column.

[Memory aids: Soh, Cah, Toa, or my favorite S: **O**scar **H**ad, C: **A** Heap, T: **O**f **A**pples]

Example 7

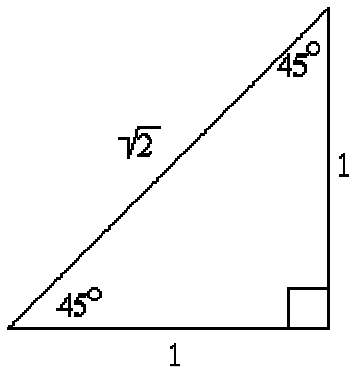
Assume $b = 12$ and $a = 5$, then find the value of each of the six trig functions of angle A.



Special Angles

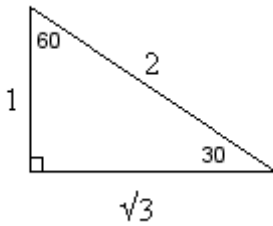
Example 8

Evaluate the six trigonometric functions for the 45° angle ($\frac{\pi}{4}$) below:

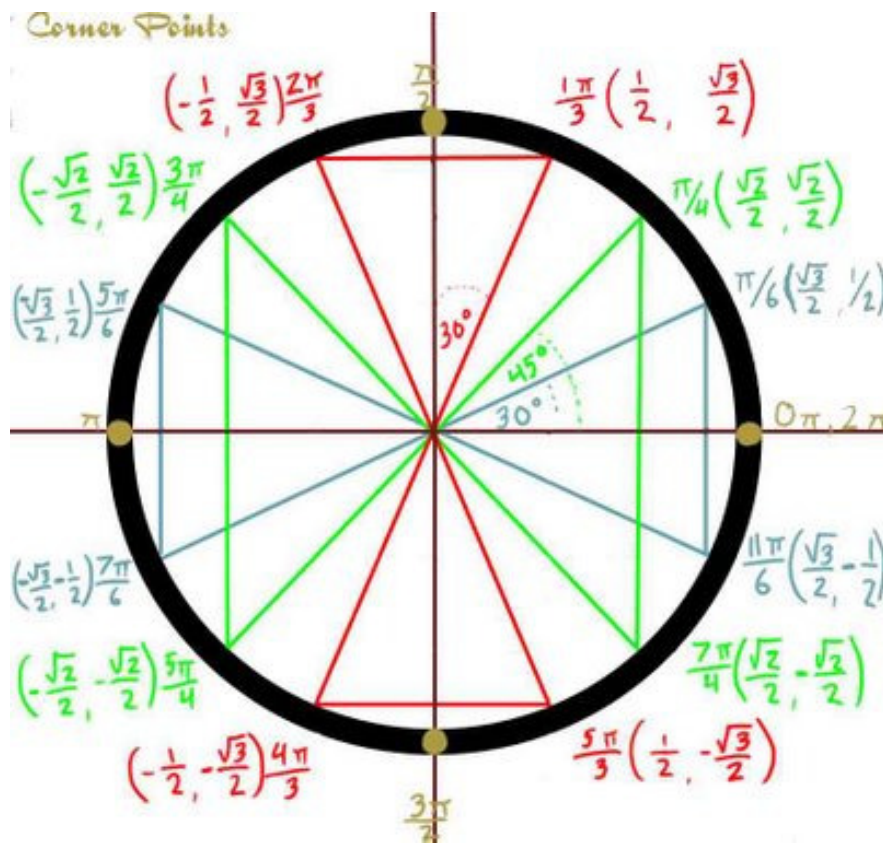


Example 9

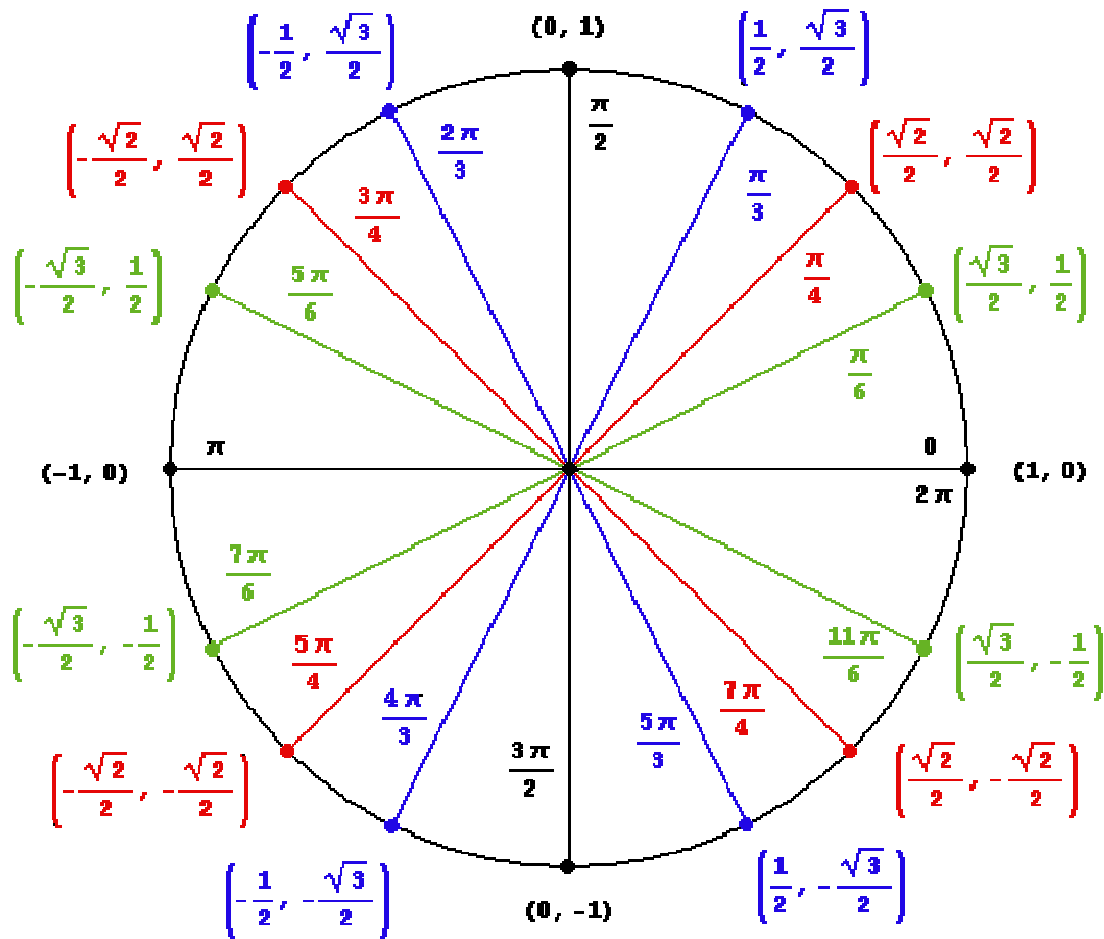
Evaluate sine, cosine, and tangent functions for the 60° or $\frac{\pi}{3}$ and the 30° or $\frac{\pi}{6}$ below:



I will **require you to memorize** the sine and cosine values for the 16 angles shown below on the **Unit Circle**:



Below is another version that is computer drawn:



Fundamental Identities

The reciprocal identities:

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

The quotient identities:

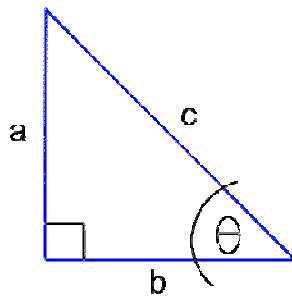
$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Example 10

Given $\sin \theta = \frac{1}{2}$ and $\cos \theta = \frac{\sqrt{3}}{2}$, find the value of each of the remaining trigonometric functions.

Recall the Pythagorean Theorem for right triangles, $a^2 + b^2 = c^2$, and consider the following triangle:



Let's divide the equation by c^2 , then we have $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$

Which is the same as $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$

Relative to θ in our drawing this leads us to perhaps the most important identity in this course:

$$\sin^2 \theta + \cos^2 \theta = 1$$

***Please note that $(\sin \theta)^2 = \sin^2 \theta$

By using the same approach but dividing by either a^2 and b^2 , we can arrive at the following identities.

The Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

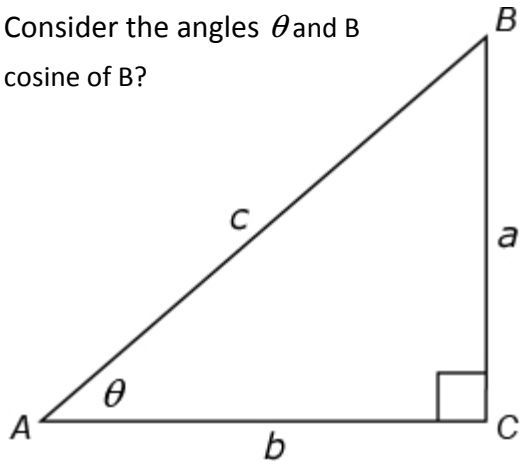
$$1 + \cot^2 \theta = \csc^2 \theta$$

Example 11

Given that $\sin \theta = \frac{3}{5}$ and that θ is an acute angle, find the value of $\cos \theta$ using the Pythagorean identity.

Trig Functions and Complements: As mentioned above, the two non 90° angles must add to 90° . This makes them complementary. This means that the cosine of one of these angles will always equal the sine of the other angle and vice versa. You will also see this pattern on the unit circle I asked you to memorize. Let's see why, consider the drawing below:

Consider the angles θ and B
cosine of B?



What is the sine of θ ? How does that compare with the
(Notice that angle B = $90^\circ - \theta$)

The Cofunction Identities (note if using radian measure use $\frac{\pi}{2}$ instead of 90°)

$$\sin \theta = \cos(90^\circ - \theta) \quad \cos \theta = \sin(90^\circ - \theta)$$

$$\tan \theta = \cot(90^\circ - \theta) \quad \cot \theta = \tan(90^\circ - \theta)$$

$$\sec \theta = \csc(90^\circ - \theta) \quad \csc \theta = \sec(90^\circ - \theta)$$

Example 12

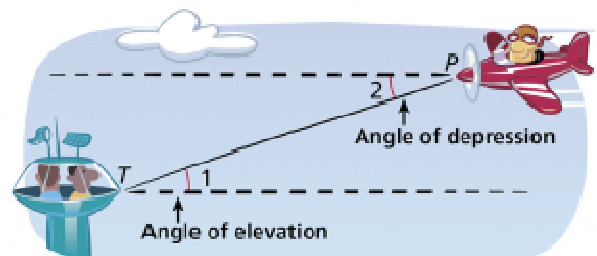
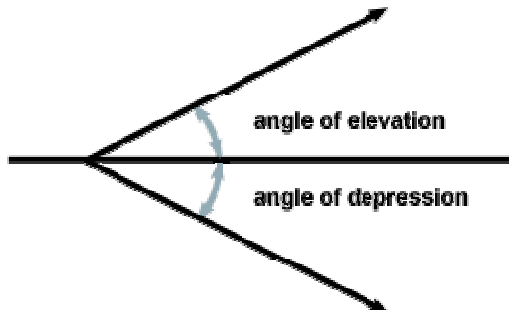
Find a cofunction with the same value as the expression given:

a. $\sin 38^\circ$

b. $\csc \frac{\pi}{3}$

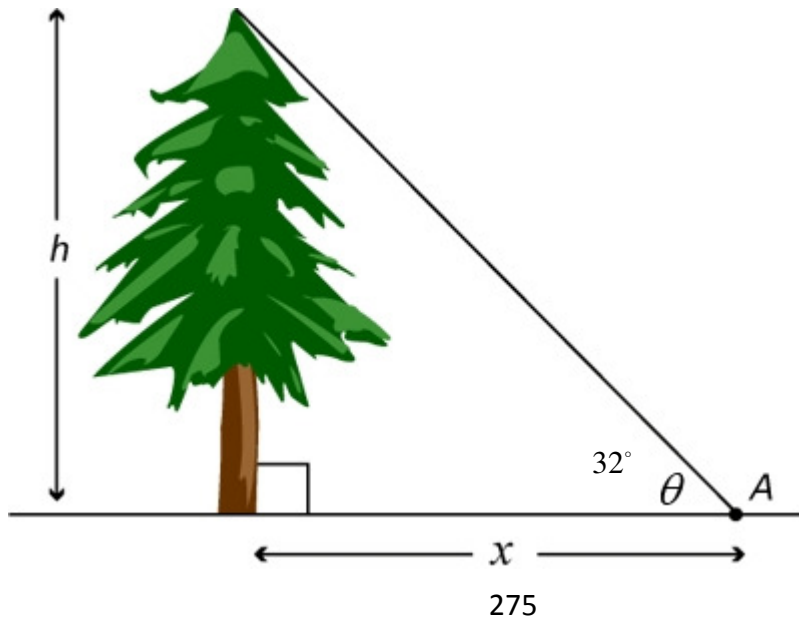
Applications

Many problems of right angle trigonometry involve the angle made with an imaginary horizontal line. The angle formed by that line and the line of sight to an object that is above the horizontal is called the **angle of elevation**. If the object is below the horizontal the angle is called the **angle of depression**.



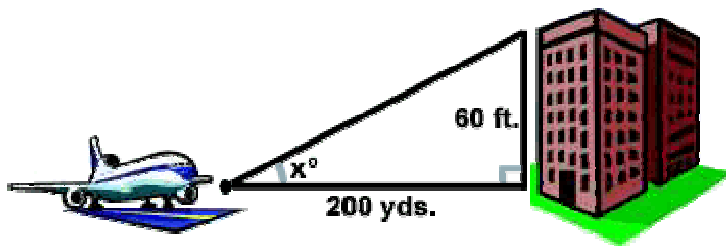
Example 13

A surveyor sights the top of a Giant Sequoia tree in California and measures the angle of elevation to be 32 degrees. The surveyor is standing 275 feet away from the base of the tree. What is the height of the tree?



Example 14

A plane must achieve a minimum angle of elevation once it reaches the end of its runway which is 200 yards from a six story building in order to clear the top of the building safely. What angle of elevation must the plane exceed in order to clear the 60 ft tall building in its path?

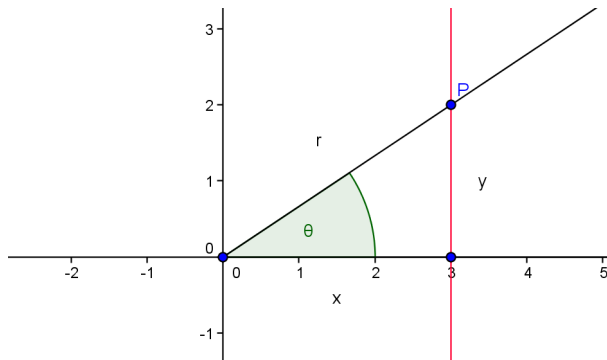


Section 5.3

Trig Functions of Any Angle

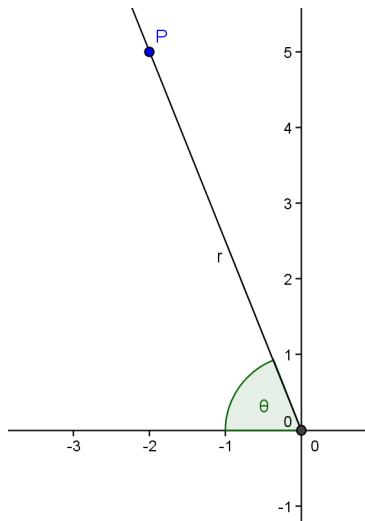
Let θ be any angle in standard position, and let $P = (x, y)$ be a point on the terminal side of θ . If $r = \sqrt{x^2 + y^2}$ is the distance from $(0, 0)$ to (x, y) , as shown below, the six trig functions of θ are defined as:

$$\begin{array}{lll} \sin \theta = \frac{y}{r} & \cos \theta = \frac{x}{r} & \tan \theta = \frac{y}{x}, x \neq 0 \\ \csc \theta = \frac{r}{y}, y \neq 0 & \sec \theta = \frac{r}{x}, x \neq 0 & \cot \theta = \frac{x}{y}, y \neq 0 \end{array}$$



Example 15

Let $P = (-2, 5)$ be a point on the terminal side of θ . Find each of the six trigonometric functions of θ .



Example 16

Find the sine function and the tangent function (if possible) at the following four quadrantal angles:

a. $\theta = 0$

b. $\theta = \frac{\pi}{2}$

c. $\theta = \pi$

d. $\theta = \frac{3\pi}{2}$

Signs of the Trigonometric Functions

Since the six trigonometric function all depend on the quantities x , y , and r and since r is **always positive**, the sign of a trigonometric function depends upon the quadrant in which θ lies. For example, in quadrant II we have angles of measures that are in the following set $\{\theta \mid \theta \in (90^\circ, 180^\circ)\}$. This means that the sines of all of those angles are positive (since all y 's are positive in quad II), the cosines of all of those angles are negative (since all x 's are negative in quad II), and finally the tangents in quadrant II are always negative (since x and y have different signs).

Example 17

If $\tan \theta > 0$ and $\cos \theta < 0$, name the quadrant in which angle θ lies.

Example 18

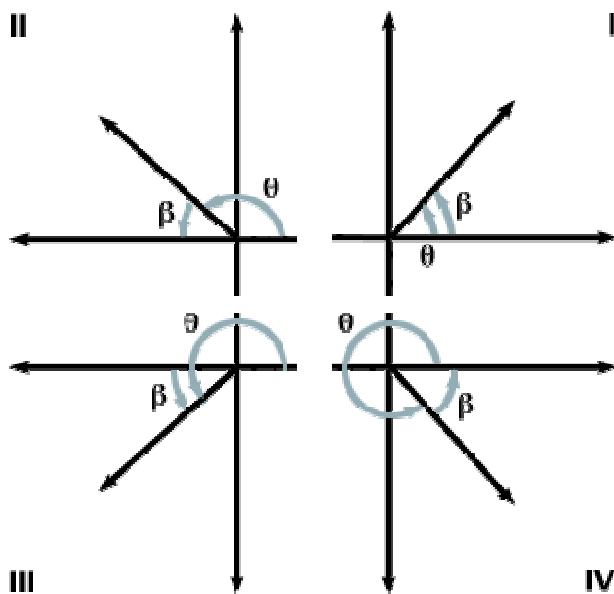
Given $\tan \theta = \frac{-2}{3}$ and $\cos \theta > 0$, find $\cos \theta$ and $\csc \theta$.

Reference Angles

We will often evaluate the trigonometric functions of positive angles greater than 90° and all negative angles by making use of a positive acute angle. This angle is called a *reference angle*.

Definition of a Reference Angle

Let θ be a non-acute angle in standard position that lies in a quadrant. Its **reference angle** is the positive acute angle θ' formed by the terminal side of θ and the x -axis.



If $90^\circ < \theta < 180^\circ$, then $\beta = 180^\circ - \theta$

If $180^\circ < \theta < 270^\circ$,
then $\beta = \theta - 180^\circ$

If $270^\circ < \theta < 360^\circ$, then $\beta = 360^\circ - \theta$

Example 19

Find the reference angle, β , for each of the following angles:

- a. $\theta = 345^\circ$ b. $\theta = \frac{5\pi}{6}$ c. $\theta = -135^\circ$ d. $\theta = 2.5$

Using Reference Angles to Evaluate Trigonometric Functions

The value of a trigonometric function of any angle θ is found as follows:

1. Find the appropriate reference angle, β .
2. Determine the required function value for β (i.e.-sine, cos, tan, ...).
3. Use the quadrant that θ lies in to determine the appropriate sign for the answer from step 2.

Example 20

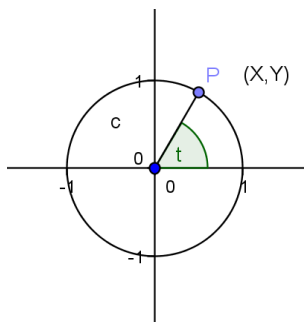
Use a reference angle to determine the exact value of each of the following trig functions:

- a. $\sin 135^\circ$ b. $\cos \frac{4\pi}{3}$ c. $\cot \left(\frac{-\pi}{3} \right)$

Section 5.4

Trigonometric Functions of Real Numbers; Periodic Functions

The unit circle we have been working with has the equation $x^2 + y^2 = 1$, if we were to solve this equation of a circle for y this equation does not form a function of x (clearly, it would fail the vertical line test). However, recall our arc length formula $s = r\theta$, for the unit circle, r is equal to one. This means our arc length formula will give us this simpler relationship: $s = 1\theta$ (or just $s = \theta$). The length of the intercepted arc is equal to the radian measure of the angle. Now let's change up the notation a bit. We will call the angle t instead of θ in this section. Now consider the diagram below:



For each real number value for t , there is a corresponding point $P = (x,y)$ on the unit circle. We can now define the cosine function at t to be the x -coordinate of P and the sine function at t to be the y -coordinate of P . Namely: $x = \cos t$ and $y = \sin t$.

The Definitions of the Trigonometric Functions in Terms of a Unit Circle

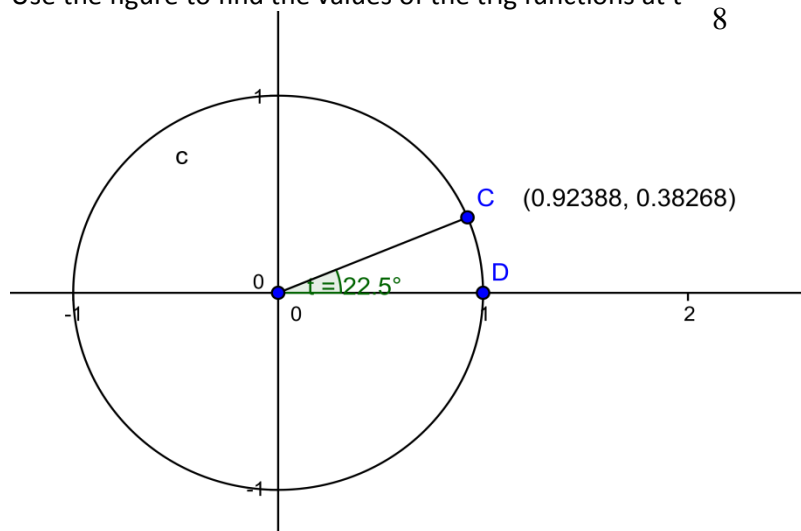
If t is a real number and $P = (x, y)$ is a point on the unit circle that corresponds to t , then:

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, x \neq 0$$

$$\csc t = \frac{1}{y}, y \neq 0 \qquad \sec t = \frac{1}{x}, x \neq 0 \qquad \cot t = \frac{x}{y}, y \neq 0$$

Example 21

Use the figure to find the values of the trig functions at $t = \frac{\pi}{8}$



Domain and Range of the Sine and Cosine Functions

Let's consider the unit circle definitions of sine and cosine, namely: $\cos t = x$ and $\sin t = y$. Since t corresponds to an arc length along the unit circle it can be any real number. This means that the domain for these two functions must be all real numbers. Now, because the radius of the unit circle is one, we have restrictions on the values that are possible for both y and x , so the ranges will be given by:

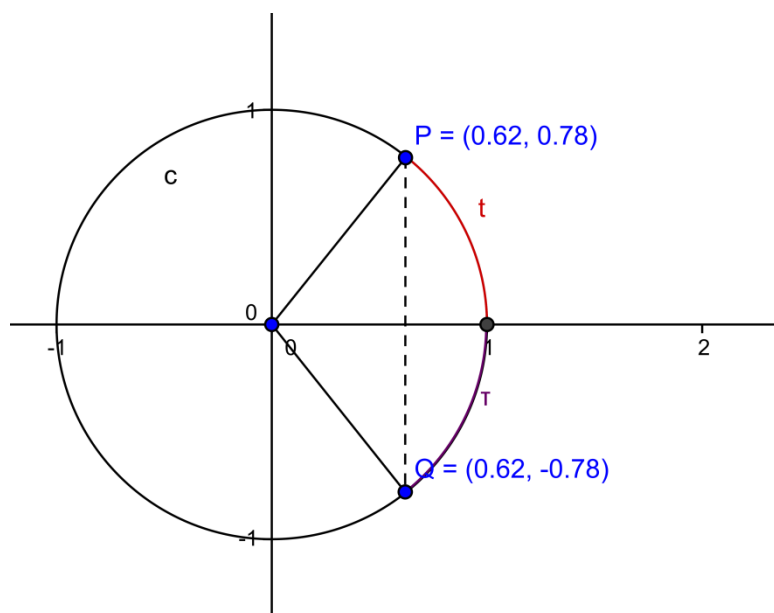
$[-1, 1]$. That is $-1 \leq \cos t \leq 1$, and $-1 \leq \sin t \leq 1$.

Even and Odd Trigonometric Functions

Recall that an even function is a function that has the following property: $f(-x) = f(x)$

Also, recall that an odd function is a function that has the following property: $f(-x) = -f(x)$

Now, consider two points on the unit circle $P : (\cos t, \sin t)$ and $Q : (\cos(-t), \sin(-t))$, I have created a drawing of two such points:



Notice how the x value is the same for both points in spite

of the fact that the point Q uses the angle $-t$, but you can also see that the y values have the opposite sign of each other.

This diagram helps us to see that $\cos t = \cos(-t)$ which means that cosine is an even function, and sine is an odd function. $\sin(-t) = -\sin t$

Even and Odd Trigonometric Functions

The cosine and secant functions are even.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are odd.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

Example 22

Find the exact value of a. $\cos(-45^\circ)$ and b. $\tan(-\frac{\pi}{3})$.

Periodic Functions

A function f is periodic if there exists a positive number p such that $f(t+p) = f(t)$ for all t in the domain of f . The smallest number p for which f is periodic is called the **period** of f .

Periodic Properties of the Sine and Cosine Functions

$$\sin(t + 2\pi) = \sin t \qquad \cos(t + 2\pi) = \cos t$$

The sine and cosine functions are periodic functions and have period 2π .

Example 23

Find the exact value of: a. $\tan 420^\circ$ and b. $\sin \frac{9\pi}{4}$

Periodic Properties of the Tangent and Cotangent Functions

$$\tan(t + \pi) = \tan t \qquad \cot(t + \pi) = \cot t$$

The tangent and cotangent functions are periodic functions and have period π .

Repetitive Behavior of the Sine, Cosine, and Tangent Functions

For any integer n and real number t ,

$$\sin(t + 2\pi n) = \sin t, \quad \cos(t + 2\pi n) = \cos t, \quad \text{and} \quad \tan(t + \pi n) = \tan t$$

Section 5.5

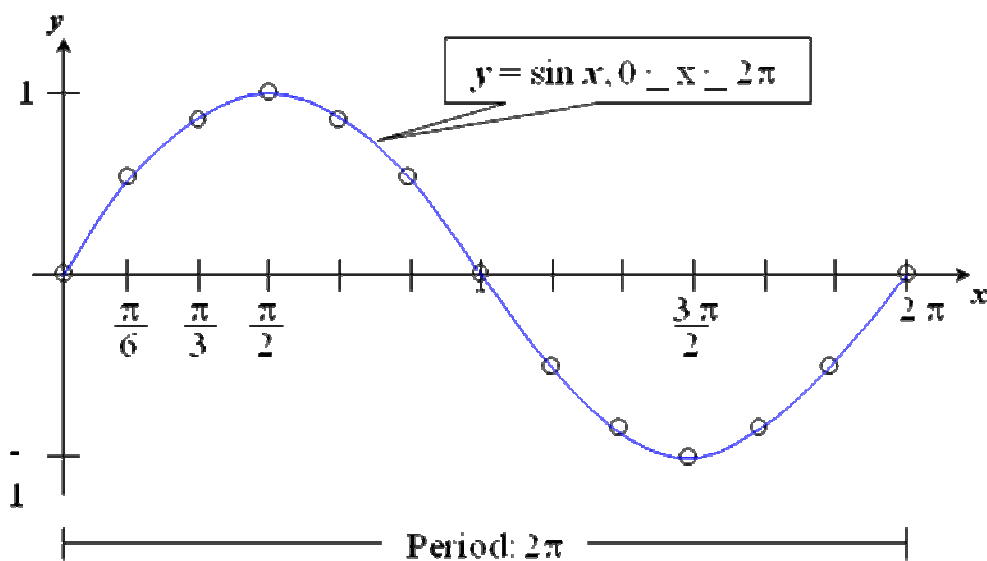
Graphs of Sine and Cosine Functions

The Graph of $y = \sin x$

We can graph the trig functions in the rectangular coordinate system by plotting points whose coordinates satisfy the function. Because the period of the sine function is 2π , we will graph the function on the interval $[0, 2\pi]$.

Table of values (x, y) on $y = \sin x$

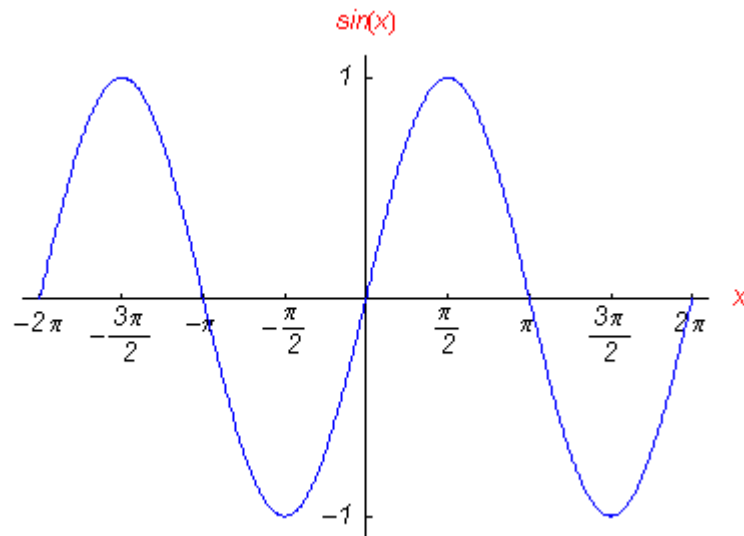
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0



Below is a graph of the sine curve which uses two periods instead of just the one we have above. From the graph of the sine wave we can see the following things:

- The domain is the set of all real numbers.

- The range consists of all numbers from $[-1, 1]$
- The period is 2π .
- This function is an odd function which can be seen from a graph by observing symmetry with respect to the origin.



Graphing Variations of $y = \sin x$

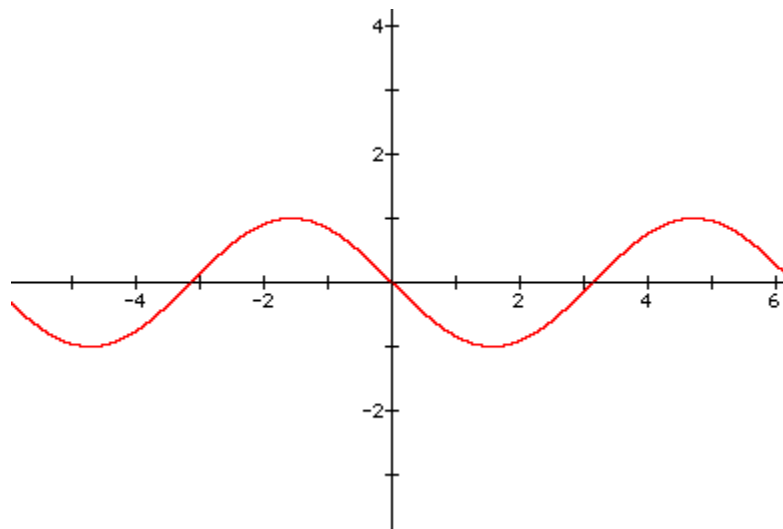
To graph variations of $y = \sin x$ by hand, it is helpful to find x-intercepts, maximum points, and minimum points. There are three intercepts, so we can find these five points by using the following scheme. Let x_1 = the x value where the cycle begins. Then for all i in $[2,5]$ the i th

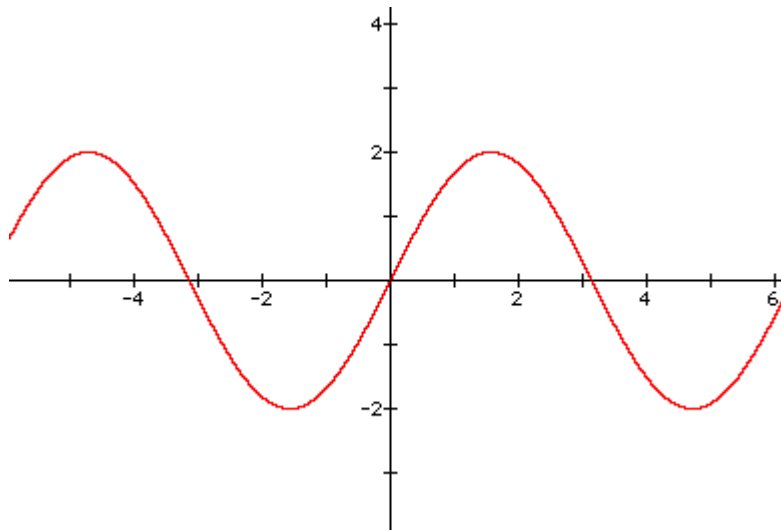
point can be found using: $x_i = x_{i-1} + \frac{\text{period}}{4}$.

We can stretch or shrink the graph by adding a coefficient other than one to the sine graph:

$$y = A \sin x$$

This is the graph of $y = \sin x$, compare it to the graph below of $y = 2 \sin x$:





In general the graph of $y = A \sin x$ ranges between $-|A|$ and $|A|$. We call A the **amplitude** of the graph. If it is $|A| > 1$ the graph is stretched, if $|A| < 1$ it is shrunk.

Graphing Variations of $y = \sin x$

- **Identify the amplitude and the period.**
- **Find the values of x for the five key points – the three x -intercepts, the maximum point, and the minimum point. Start with the value of x where the cycle begins and add quarter-periods – that is, $\text{period}/4$ – to find successive values of x .**
- **Find the values of y for the five key points by evaluating the function at each value of x from step 2.**
- **Connect the five key points with a smooth curve and graph one complete cycle of the given function.**
- **Extend the graph in step 4 to the left or right as desired.**

Example 24

Determine the amplitude of $y = 1/2 \sin x$. Then graph $y = \sin x$ and $y = 1/2 \sin x$ for

$$0 \leq x \leq 2\pi.$$

Step 1 Identify the amplitude and the period. The equation $y = 1/2 \sin x$ is of the form $y = A \sin x$ with $A = 1/2$. Thus, the amplitude $|A| = 1/2$. This means that the maximum value of y is $1/2$ and the minimum value of y is $-1/2$. The period for both $y = 1/2 \sin x$ and $y = \sin x$ is 2π .

Step 2 Find the values of x for the five key points. We need to find the three x -intercepts, the maximum point, and the minimum point on the interval $[0, 2\pi]$. To do so, we begin by dividing the period, 2π , by 4.

$$\text{Period}/4 = 2\pi/4 = \pi/2$$

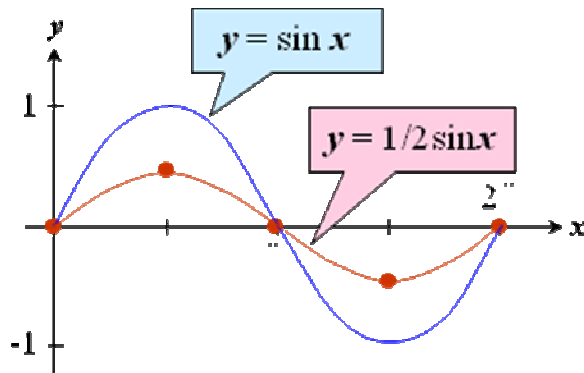
We start with the value of x where the cycle begins: $x = 0$. Now we add quarter periods, $\pi/2$, to generate x -values for each of the key points. The five x -values are

$$x = 0, x = \pi/2, x = \pi, x = 3\pi/2, x = 2\pi$$

Step 3 Find the values of y for the five key points. We evaluate the function at each value of x from step 2.

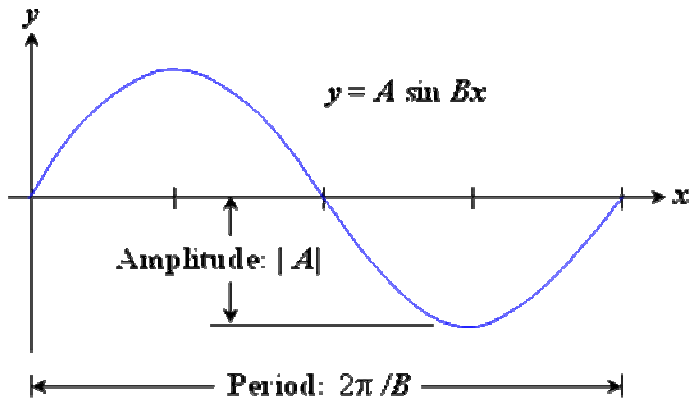
$$(0,0), (\pi/2, 1/2), (\pi,0), (3\pi/2, -1/2), (2\pi, 0)$$

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The five key points for $y = 1/2\sin x$ are shown below. By connecting the points with a smooth curve, the figure shows one complete cycle of $y = 1/2\sin x$. Also shown is graph of $y = \sin x$. The graph of $y = 1/2\sin x$ shrinks the graph of $y = \sin x$.



Amplitudes and Periods

The graph of $y = A \sin Bx$ has amplitude $= |A|$ and period $= \frac{2\pi}{B}$.



Example 25

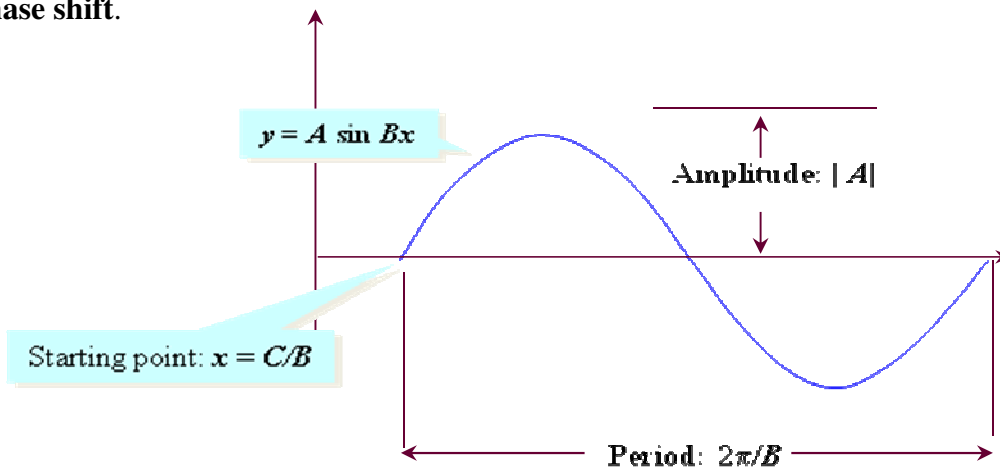
Determine the amplitude and period of $y = 3\sin 2x$, then graph the function for $0 \leq x \leq 2\pi$.

Steps

1. Determine the amplitude and the period using the form $y = A \sin Bx$.
2. Divide the period by four and find the five key points using $x_i = x_{i-1} + \frac{\text{period}}{4}$.
3. Get your resulting y-values after plugging in the above x-values.
4. Extend the graph as you wish

The Graph of $y = A \sin(Bx - C)$

The graph of $y = A \sin(Bx - C)$ is obtained by horizontally shifting the graph of $y = A \sin Bx$ so that the starting point of the cycle is shifted from $x = 0$ to $x = \frac{C}{B}$. The number $\frac{C}{B}$ is called the **phase shift**.



Example 26

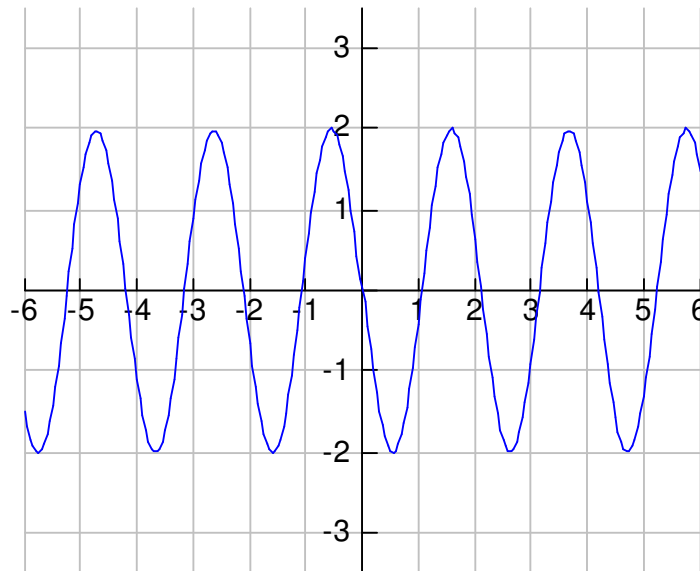
Determine the amplitude, period, and phase shift of $y = 2\sin(3x - \pi)$

Solution: $y = A \sin(Bx - C)$

Amplitude = $|A| = 2$

period = $2\pi/B = 2\pi/3$

phase shift = $C/B = \pi/3$



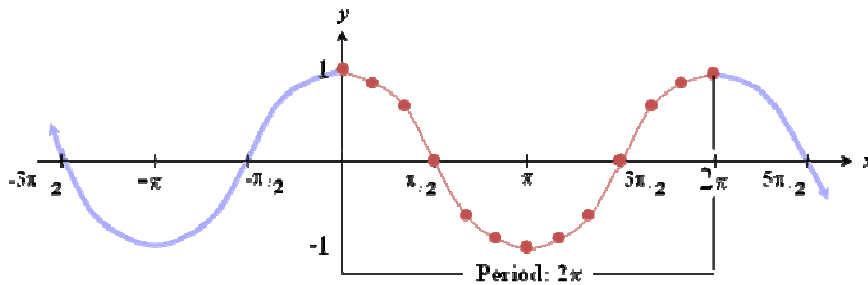
$y = 2\sin(3x - \pi)$

The Graph of $y = A\cos Bx$

We graph $y = \cos x$ by listing some points on the graph. Because the period of the cosine function is 2π , we will concentrate on the graph of the basic cosine curve on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion.

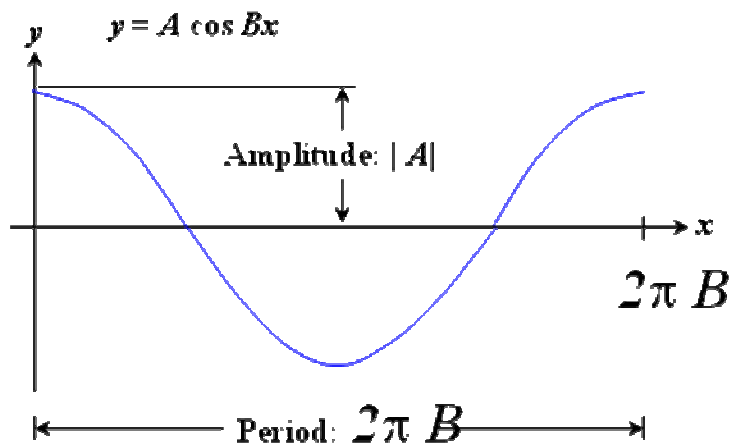
Below is a graph of the cosine curve. From the graph we can see the following things:

- The domain is the set of all real numbers.
- The range consists of all numbers from $[-1, 1]$
- The period is 2π .
- This function is an even function which can be seen from a graph by observing symmetry with respect to the y -axis.



Note the similarity between the graphs of Sine and Cosine. In fact, they are related by the following identity: $\cos x = \sin\left(x + \frac{\pi}{2}\right)$

Like the sine graph, the graph of $y = A \cos Bx$ has amplitude = $|A|$ & period = $2\pi/B$.



Example 27

Determine the amplitude and period of $y = -3\cos\frac{\pi}{2}x$. Then graph the function for $[-4, 4]$.

Steps

1. Determine the amplitude and the period using the form $y = A \cos Bx$.
2. Divide the period by four and find the five key points using $x_i = x_{i-1} + \frac{\text{period}}{4}$.
3. Get your resulting y-values after plugging in the above x-values.
4. Extend the graph as you wish.

The Graph of $y = A \cos(Bx - C)$

The graph of $y = A \cos(Bx - C)$ is obtained by horizontally shifting the graph of $y = A \cos Bx$ so that the starting point of the cycle is shifted from $x = 0$ to $x = \frac{C}{B}$. The number $\frac{C}{B}$ is called the **phase shift**. Amplitude = $|A|$ & the period = $2\pi/B$.

Remember! In the example above, **C is not negative**. In the general form, there is subtraction in the parentheses. Therefore, **C is positive**, which makes the function's **phase shift to the right**.

Example 28

Determine the amplitude, period, and phase shift of $y = \frac{1}{2} \cos(4x + \pi)$. Then graph one period of the function.

Vertical shifts of the Sinusoidal Graphs

Consider the following forms: $y = A \sin(Bx - C) + D$ and $y = A \cos(Bx - C) + D$

The constant D causes vertical shifts in the graphs of the functions. This will change the maximums and minimums so that the maximum becomes $D + |A|$ and the minimum becomes $D - |A|$.

Example 29

Graph one period of $y = \frac{1}{2} \cos x - 1$

Section 5.6

Graphs of Other Trigonometric Functions

The Graph of $y = \tan x$

The graph of tangent is very different from the graphs of sine and cosine. Here are some properties of the tangent function we should consider before graphing:

- The period is π . It is only necessary to graph tangent over an interval of length π . After this the graph just repeats.
- The tangent function is an odd function: $\tan(-x) = -\tan x$. The graph is therefore symmetric with respect to the origin.
- The tangent function is undefined at $\frac{\pi}{2}$. Therefore, the graph will have a vertical asymptote at $x = \frac{\pi}{2}$.

We obtain the graph of $y = \tan x$ using some points on the graph and origin symmetry. The table below lists some tangent values over the interval $\left[0, \frac{\pi}{2}\right)$.

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$ (75°)	$\frac{17\pi}{36}$ (85°)	$\frac{89\pi}{180}$ (89°)	1.57	$\frac{\pi}{2}$
Y = tan x	0	$\frac{\sqrt{3}}{3} \approx 0.6$	1	$\sqrt{3} \approx 1.7$	3.7	11.4	57.3	1255. 8	Undefined

**Notice how as we approach 90° the graphs increases slowly at first then dramatically. This due to the vertical asymptote at $\frac{\pi}{2}$. Below are some properties of the tangent function and its graph:

Period: π

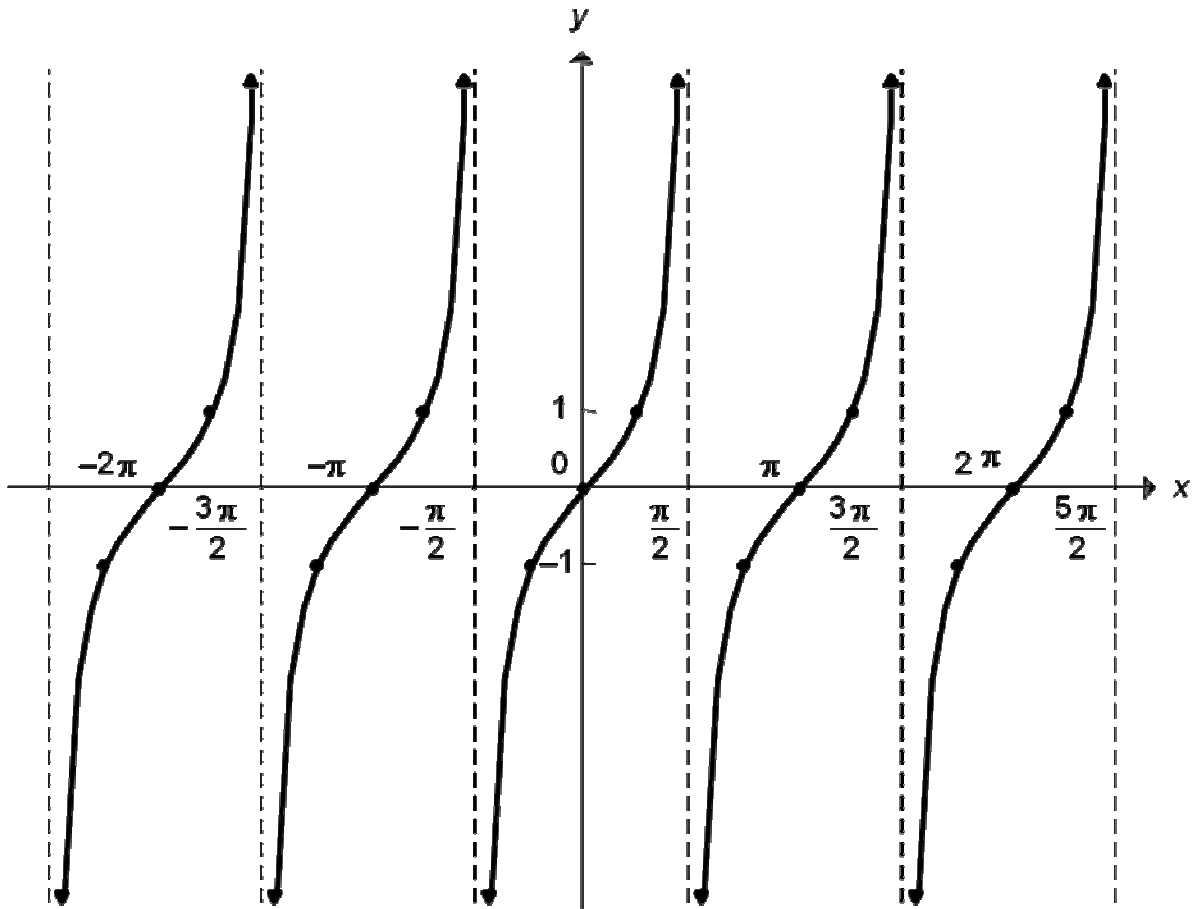
Domain: All real numbers except $\frac{\pi}{2} + k\pi$, where k an integer

Range: All real numbers

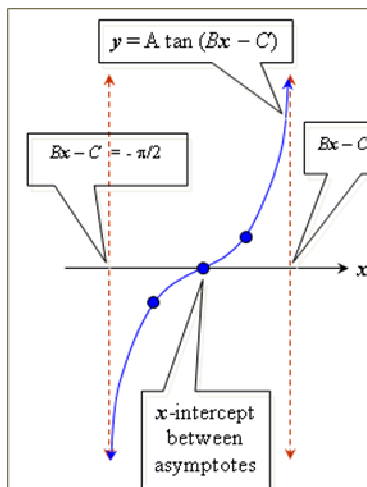
Symmetric with respect to the origin

Vertical asymptotes at all odd multiples of $\frac{\pi}{2}$

Below is the graph of $y = \tan x$:



Graphing $y = A \tan(Bx - C)$



1. Find two consecutive asymptotes by setting the variable expression in the tangent equal to $-\pi/2$ and $\pi/2$ and solving

$$Bx - C = -\pi/2 \text{ and } Bx - C = \pi/2$$
2. Identify an x-intercept, midway between consecutive asymptotes.
3. Find the points on the graph $1/4$ and $3/4$ of the way between and x-intercept and the asymptotes. These points have y-coordinates of $-A$ and A .
4. Use steps 1-3 to graph one full period of the function. Add additional cycles to the left or right as needed.

Example 30

Graph $y = 2 \tan \frac{x}{2}$ for $-\pi < x < 3\pi$

- Find two consecutive asymptotes by solving $Bx - C = -\frac{\pi}{2}$ and $Bx - C = \frac{\pi}{2}$.
- Identify an x-intercept midway between the asymptotes (average the asymptotes x's).
- Find points $\frac{1}{4}$ and $\frac{3}{4}$ of the way between consecutive asymptotes. These points have y-coordinates $-A$ and A .
- Connect these points with a smooth curve.

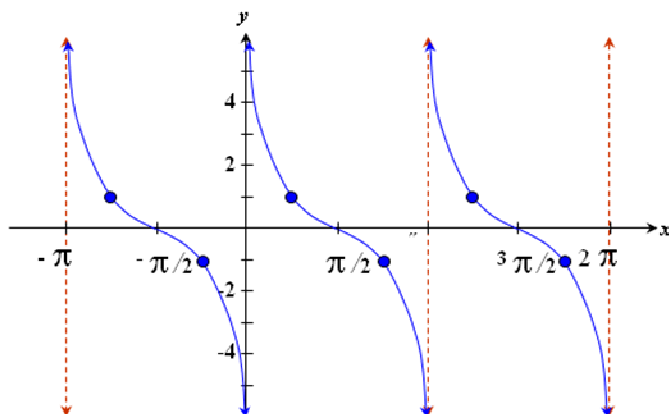
Example 31

Graph two full periods of $y = \tan\left(x + \frac{\pi}{4}\right)$

- Find two consecutive asymptotes by solving $Bx - C = -\frac{\pi}{2}$ and $Bx - C = \frac{\pi}{2}$.
- Identify an x-intercept midway between the asymptotes (average the asymptotes x's).
- Find points $\frac{1}{4}$ and $\frac{3}{4}$ of the way between consecutive asymptotes. These points have y-coordinates $-A$ and A .
- Connect these points with a smooth curve.

The Cotangent Curve: The Graph of $y = \cot x$ and Its Characteristics

The Graph of $y = \cot x$ and Its Characteristics



Characteristics

Period: π

Domain: All real numbers except integral multiples of π

Range: All real numbers

Vertical asymptotes: at integral multiples of π

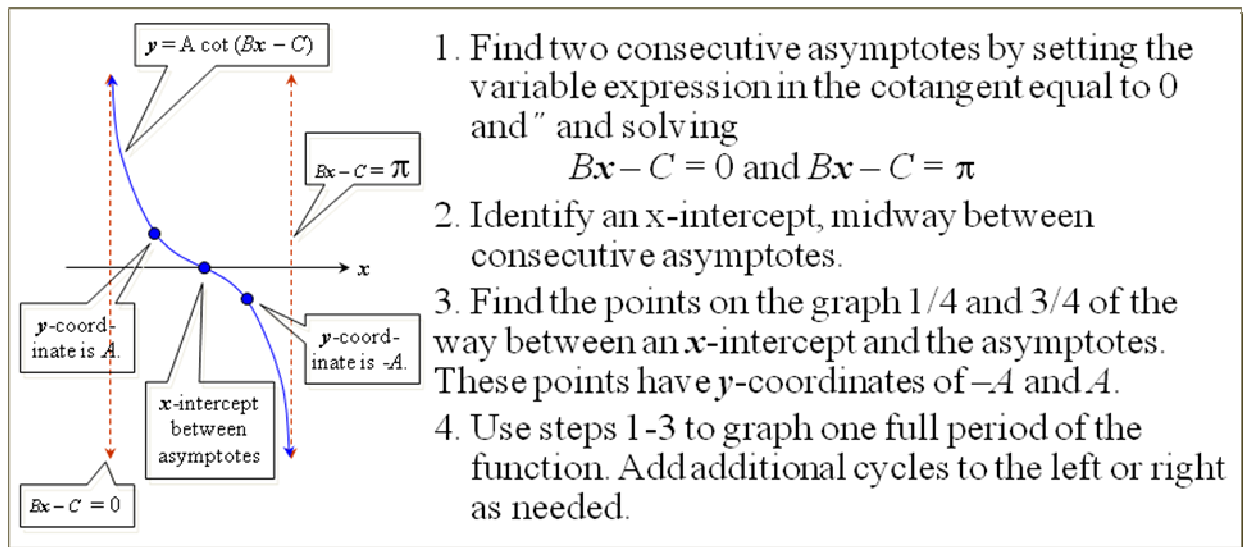
An x-intercept occurs midway between each pair of consecutive asymptotes.

Odd function with origin symmetry

Points on the graph $\frac{1}{4}$ and $\frac{3}{4}$ of the way between consecutive asymptotes have y-coordinates of -1 and 1 .

Graphing Variations of $y = \cot x$

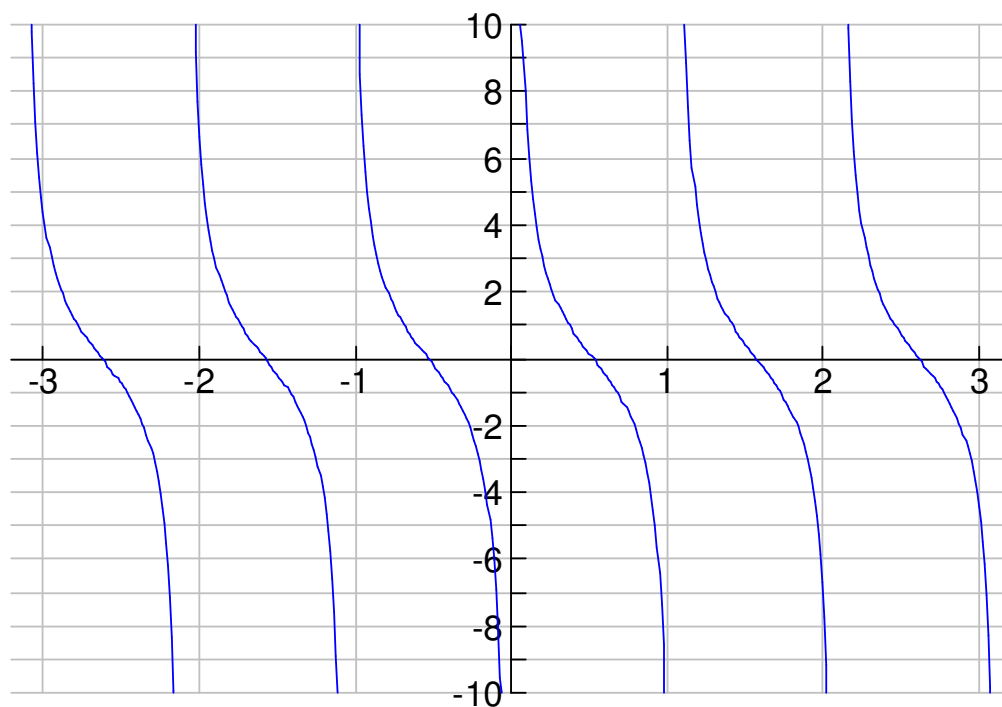
Graphing $y = A \cot(Bx - C)$



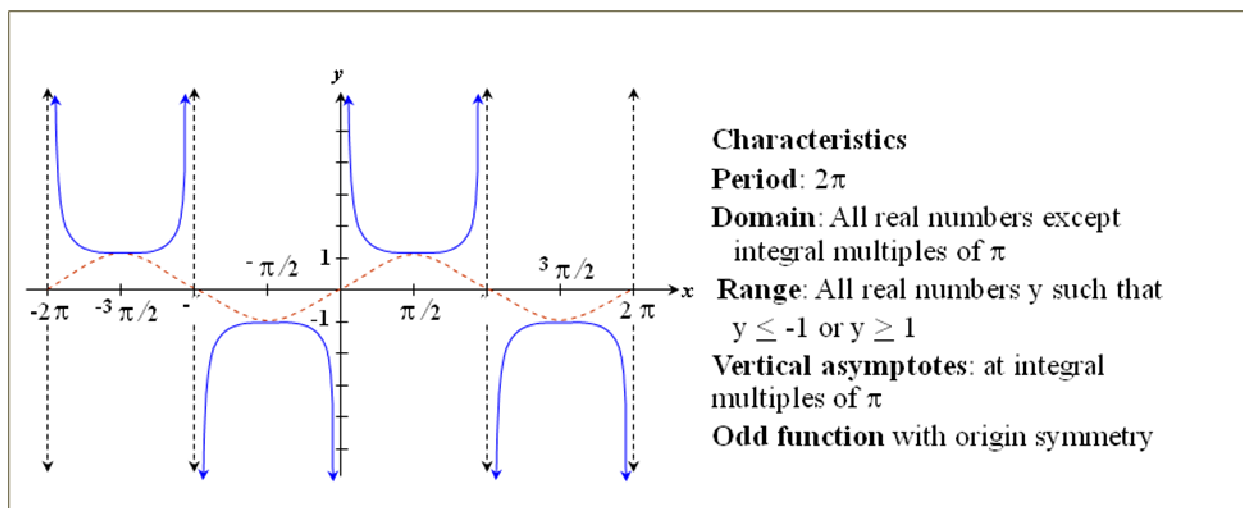
Example 32

Graph $y = 3 \cot 2x$

- Find two consecutive asymptotes by solving $Bx - C = 0$ and $Bx - C = \pi$.
- Identify an x-intercept midway between the asymptotes (average the asymptotes x's).
- Find points $1/4$ and $3/4$ of the way between consecutive asymptotes. These points have y-coordinates A and $-A$.
- Connect these points with a smooth curve.



The Cosecant Curve: The Graph of $y = \csc x$ and Its Characteristics

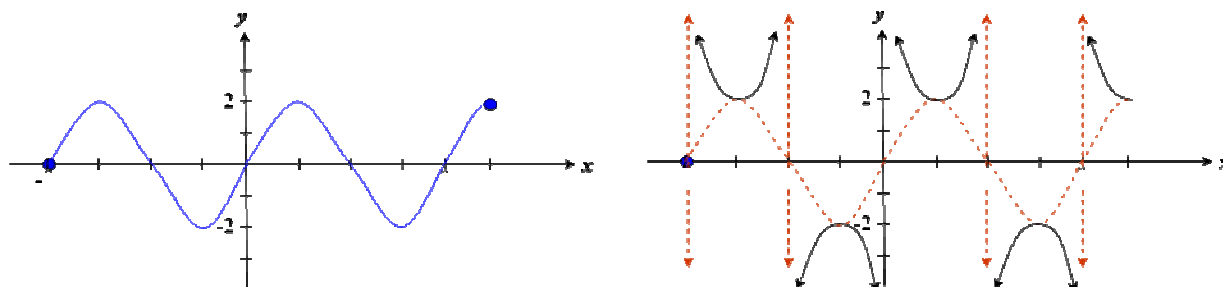


For the Cosecant and Secant curves it is helpful to graph the related Sine and Cosine curves first.

- X-intercepts on the sine curve correspond to vertical asymptotes on the cosecant curve.
- A maximum on the sine curve represents a minimum on the cosecant curve.
- A minimum on the sine curve represents a maximum on the cosecant curve.

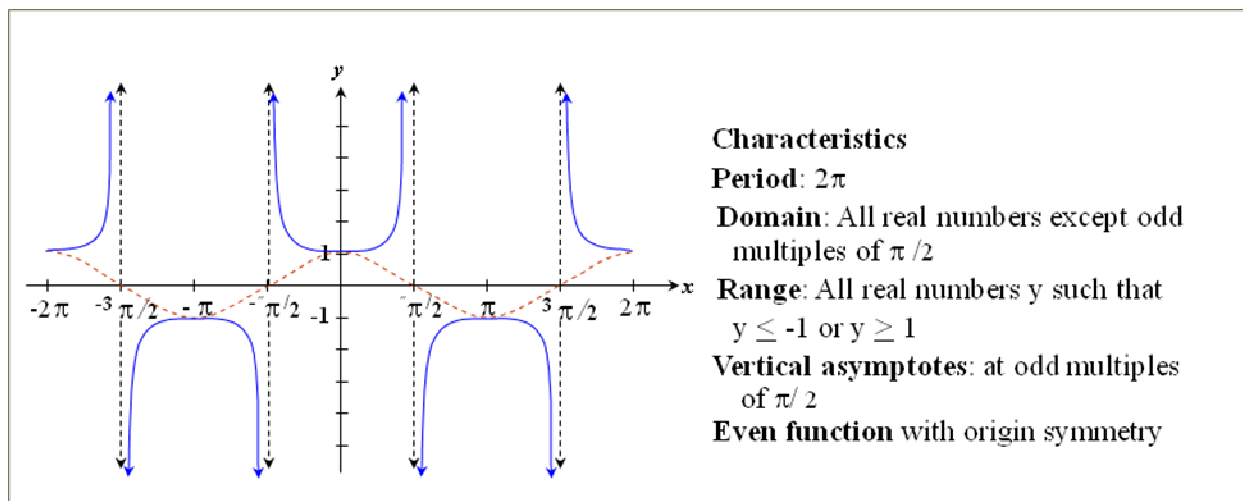
Example 33

Use the graph of $y = 2 \sin 2x$ to obtain the graph of $y = 2 \csc 2x$.



Solution The x-intercepts of $y = 2 \sin 2x$ correspond to the vertical asymptotes of $y = 2 \csc 2x$. Thus, we draw vertical asymptotes through the x-intercepts. Using the asymptotes as guides, we sketch the graph of $y = 2 \csc 2x$.

The Secant Curve: The Graph of $y = \sec x$ and Its Characteristics



- X-intercepts on the cosine curve correspond to vertical asymptotes on the secant curve.
- A maximum on the cosine curve represents a minimum on the secant curve.
- A minimum on the cosine curve represents a maximum on the secant curve.

Example 34

Graph $y = -3\sec\frac{x}{2}$ for $-\pi < x < 5\pi$ (Hint: use the graph of $y = -3\cos\frac{x}{2}$)

Section 5.7

Inverse Trigonometric Functions

Recall that in order for a function to have an inverse it must pass the horizontal line test (i.e.-it must be one-to-one). Also, the inverse function's graph will be the reflection of the original function about the line $y = x$.

Since the trig functions would fail to be one-to-one on their entire domain due to their periodic nature among other things, we must restrict their domains to be able to find their inverses.

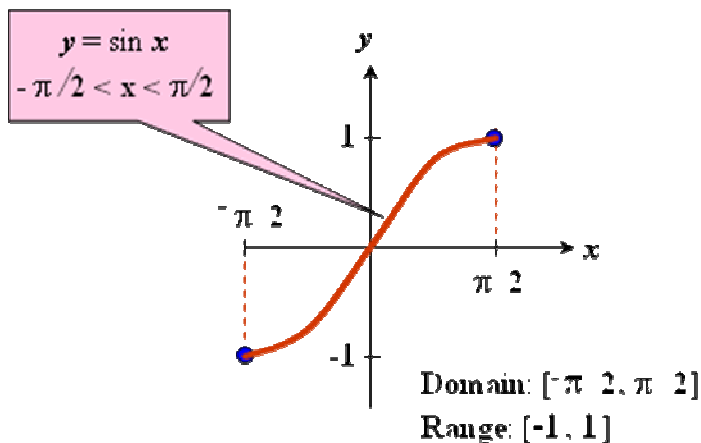
The Inverse Sine Function

The **inverse sine function**, denoted by \sin^{-1} , is the inverse of the restricted sine function:

$y = \sin x$, $-\pi/2 \leq x \leq \pi/2$. Thus,

$y = \sin^{-1} x$ means $\sin y = x$, "at what angle is the sine equal to x ?"

where $-\pi/2 \leq y \leq \pi/2$ and $-1 \leq x \leq 1$. We read $y = \sin^{-1} x$ as "y equals the inverse sine at x ."



We can graph the inverse of the sine function with the above restricted domain by simple reversing the points. For example, (x, y) is switched to (y, x) .

Finding Exact Values of $\sin^{-1}x$

- Let $\theta = \sin^{-1} x$.
- Rewrite step 1 as $\sin \theta = x$.
- Use the exact values from the unit circle to find the value of θ in $[-\pi/2, \pi/2]$ that satisfies $\sin \theta = x$.

Example 35

Find the exact value of $\sin^{-1}(1/2)$

The Inverse Cosine Function

The **inverse cosine function**, denoted by \cos^{-1} , is the inverse of the restricted cosine function

$y = \cos x, 0 \leq x \leq \pi$. Thus,

$y = \cos^{-1} x$ means $\cos y = x$,

where $0 \leq y \leq \pi$ and $-1 \leq x \leq 1$.

Example 36

Find the exact value of $\cos^{-1}(-\sqrt{3}/2)$

The Inverse Tangent Function

The **inverse tangent function**, denoted by \tan^{-1} , is the inverse of the restricted tangent function

$y = \tan x, -\pi/2 < x < \pi/2$. Thus,

$y = \tan^{-1} x$ means $\tan y = x$,

where $-\pi/2 < y < \pi/2$ and $-\infty < x < \infty$.

Example 37

Find the exact value of $\tan^{-1}\sqrt{3}$

Inverse Properties

The Sine Function and Its Inverse

$$\sin(\sin^{-1} x) = x \quad \text{for every } x \text{ in the interval } [-1, 1].$$

$$\sin^{-1}(\sin x) = x \text{ for every } x \text{ in the interval } [-\pi/2, \pi/2].$$

The Cosine Function and Its Inverse

$$\cos(\cos^{-1} x) = x \quad \text{for every } x \text{ in the interval } [-1, 1].$$

$$\cos^{-1}(\cos x) = x \quad \text{for every } x \text{ in the interval } [0, \pi].$$

The Tangent Function and Its Inverse

$$\tan(\tan^{-1} x) = x \quad \text{for every real number } x$$

$$\tan^{-1}(\tan x) = x \text{ for every } x \text{ in the interval } (-\pi/2, \pi/2).$$

Example 38

Find the exact values if possible of:

a. $\cos(\cos^{-1} 0.6)$ b. $\sin^{-1}\left(\sin \frac{3\pi}{2}\right)$ c. $\cos(\cos^{-1} 2\pi)$

Example 39

Find the exact value of $\cos\left(\tan^{-1} \frac{5}{12}\right)$.

Example 40

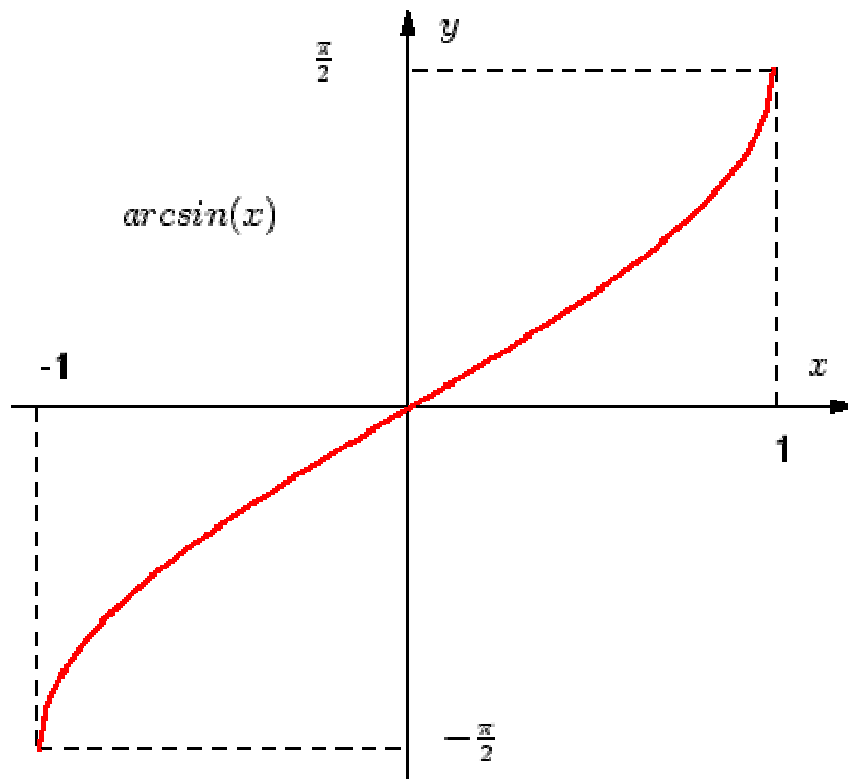
Find the exact value of $\cot\left(\sin^{-1} \frac{-1}{3}\right)$.

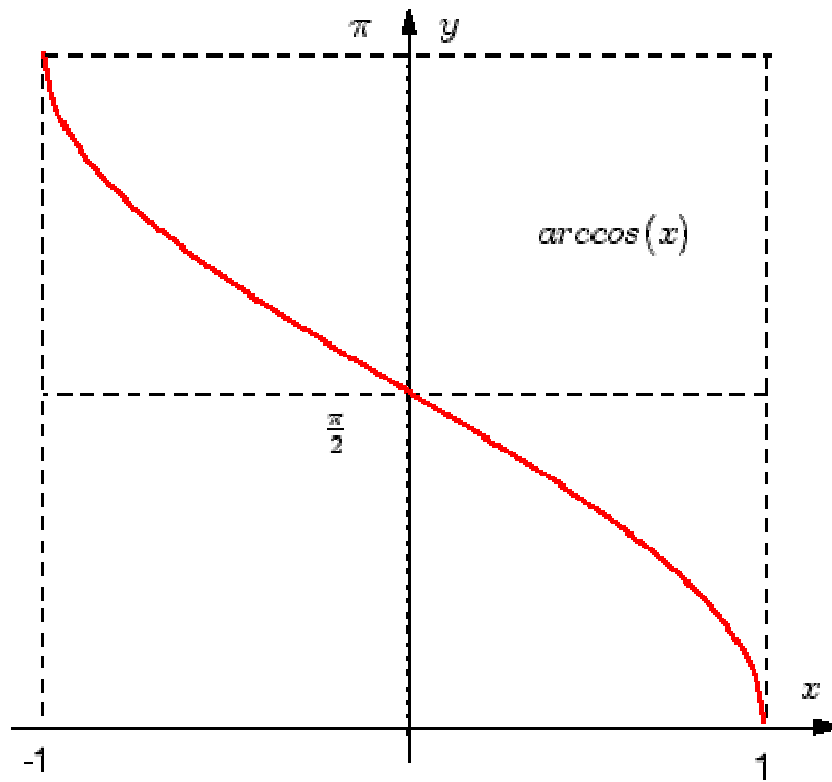
Example 41

Calculators and the Inverse Trigonometric Functions

Use your calculator to find $\sin^{-1} \frac{1}{4}$ and $\tan^{-1}(-9.65)$ in radian mode.

Graphs of the Three Basic Trigonometric Functions





Arc tangent

