# Planar Complex Vector Fields with Homogeneous Singular Points

## ICMC Summer Meeting on Differential Equations


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$$L = A(x,y)\partial_x + B(x,y)\partial_y$$

with $A,\ B\ \in C^\infty(\mathbb{R}^2\backslash 0,\ \mathbb{C})$ and homogeneous of degree $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 1$:

$$A(tx,ty) = t^\lambda A(x,y),\ \ B(tx,ty) = t^\lambda B(x,y)\ \ \forall (x,y) \in \mathbb{R}^2\backslash 0,\ \ t > 0.$$

$L$ has a singular point at 0.
Solvability of $Lu = f$ is an open set $\Omega \ni 0$ depends heavily on number theoretic properties of an associated pair $(\lambda,\mu)$:

- $\lambda$ homogeneity degree;
- $\mu$ invariant given by

$$\mu = \frac{1}{2\pi}\int_{|z|=1} \frac{xA+yB}{xB-yA}\,\frac{dz}{z},\quad z = x+iy$$

Results based on two papers

1. A.M. *Solvability of complex vector fields with homogeneous singularities*, Complex Var. Elliptic Eq. (2017)
   Among results:
   - If $\mu = i\beta \in i\mathbb{R}^*$ and $u$ is a distribution solution of $Lu = 0$ in an annulus $\{r^2 \leq x^2 + y^2 \leq R^2\}$ with $R > re^{2\pi|\beta|}$, then $u$ extends as a distribution solution to $\mathbb{R}^2$
   - If $f \in C^{\varpi}$ at 0, the equation $Lu = f$ has a solution if the pair $(\lambda, \mu)$ is nonresonant.
2. C. Campana, P. Dattori, A.M. *A class of planar vector fields with homogeneous singular points: Solvability and boundary value problems*, J. Diff. Eq. (2018)
   Solvability studied through integral operator

$$T_Z f(x,y) = \frac{1}{2\pi i} \int_{\Omega} \frac{f(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)} d\xi d\eta$$

# Preliminaries

$x = r\cos\theta\,,\ y = r\sin\theta\,,\ A = r^{\lambda}a(\theta)\,,\ B = r^{\lambda}b(\theta)\,,$
$a, b \in C^{\infty}(\mathbb{S}^1, \mathbb{C})$

$$L = A\partial_x + B\partial_y = r^{\lambda-1}\left(p(\theta)\partial_\theta - iq(\theta)r\partial_r\right) = r^{\lambda-1}L_0$$

$$p(\theta) = b(\theta)\cos\theta - a(\theta)\sin\theta\,,\ \ q(\theta) = i\left(a(\theta)\cos\theta + b(\theta)\sin\theta\right)\ .$$

Set of nonellipticity (union of ray):

$$\begin{aligned}\Sigma &= \{(x,y)\in\mathbb{R}^2;\ L\wedge\overline{L} = 0\} = \{(x,y)\in\mathbb{R}^2;\ \mathrm{Im}(A\overline{B}) = 0\}\\ &= \{(r,\theta)\in[0,\infty)\times\mathbb{S}^1;\ \mathrm{Re}(q\overline{p}) = 0\} = [0,\infty)\times\Sigma_0\end{aligned}$$

Hypotheses:

- $\Sigma$ has an empty interior in $\mathbb{R}^2$ ($\Longleftrightarrow$ $\Sigma_0$ has an empty interior in $\mathbb{S}^1$);
- $L\wedge(x\partial_x + y\partial_y) \neq 0$ everywhere in $\mathbb{R}^2\backslash 0$ ($\Longleftrightarrow$ $p(\theta)\neq 0\ \ \forall\theta\in\mathbb{S}^1$)

$L$ is locally solvable at $m \in \mathbb{R}^2$ if $\exists V \underset{open}{\subset} U \underset{open}{\subset} \mathbb{R}^2$, $m \in V$, such that $\forall f \in C^\infty(U)$ equation $Lu = f$ has a solution $u \in \mathcal{D}'(V)$.
$L$ is locally solvable at each point $m \in \mathbb{R}^2 \backslash \Sigma$. $L$ is locally solvable at $m_0 \in \Sigma \backslash 0 \iff L$ satisfies condition $(\mathcal{P})$ at $m_0$ (Im$(A\overline{B})$ does not change sign in a neighborhood of $m_0$; equivalently Re$(q\bar{p})$ does not change sign near $\theta_0 \in \mathbb{S}^1$).
Note: If $L$ satisfies condition $(\mathcal{P})$ at $m_0 \in \Sigma \backslash 0$, then hypothesis $L \wedge (x\partial_x + y\partial_y) \neq 0$ implies that $L$ is hypoelliptic at each point on the ray containing $m_0$.
To $L = A\partial_x + B\partial_y = r^{\lambda-1}(p(\theta)\partial_\theta - iq(\theta)r\partial_r)$ associate $\mu \in \mathbb{C}$

$$\mu = \frac{1}{2\pi} \int_{|z|=1} \frac{xA + yB}{xB - yA} \frac{dz}{z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{q(\theta)}{p(\theta)} d\theta \,.$$

We can assume $\mathrm{Re}(\mu) \geq 0$. If $L$ satisfies condition $(\mathcal{P})$ in $\mathbb{R}^2 \backslash 0$, then $\mathrm{Re}(\mu) > 0$.

# First Integrals

$F$ is a first integral of $L$ in $\Omega \underset{open}{\subset} \mathbb{R}^2$ if $LF = 0$ and $dF \neq 0$ in $\Omega$.

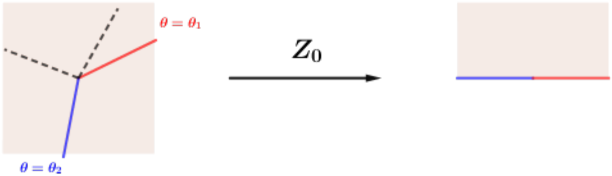
- **<u>Case:</u>** $\mu = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{q(\theta)}{p(\theta)} d\theta = 0$

$m(\theta) = \displaystyle\int_0^{\theta} \frac{q(s)}{p(s)} ds$. $m$ is $2\pi$-periodic and $\mathrm{Re}(m) \not\equiv 0$.

$\mathrm{Re}(m(\theta_1)) = \min \mathrm{Re}(m)$; $\mathrm{Re}(m(\theta_2)) = \max \mathrm{Re}(m)$. Define

$\sigma = \dfrac{\pi}{\mathrm{Re}(m(\theta_2)) - \mathrm{Re}(m(\theta_1))}$ and $\phi(\theta) = \sigma(m(\theta) - m(\theta_1))$.

$Z_0(r,\theta) = r^{\sigma} e^{i\phi(\theta)}$: First integral of $L$ and $Z_0(\mathbb{R}^2) = \mathbb{C}^+$.



- **Case:** $\mu = \frac{1}{2\pi}\int_0^{2\pi} \frac{q(\theta)}{p(\theta)} d\theta \neq 0$

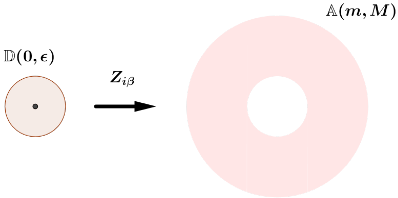
$$c_j = \frac{1}{2\pi}\int_0^{2\pi} \frac{q(\theta)}{p(\theta)} e^{ij\theta} d\theta; \quad \phi(\theta) = \sum_{j\neq 0} \frac{c_j}{ij\mu} e^{ij\theta} = \phi_1(\theta) + i\phi_2(\theta)$$

$Z_\mu(r,\theta) = r^{1/\mu} e^{i(\theta+\phi(\theta))}$ is a $C^\infty$ first integral of $L$ in $\mathbb{R}^2 \backslash 0$.

- If $\mathrm{Re}(\mu) > 0$, then $Z_\mu(\mathbb{R}^2) = \mathbb{C}$ and if $L$ satisfies condition $(\mathcal{P})$, $Z$ is a global homeomorphism.
- If $\mathrm{Re}(\mu) = 0$, $\mu = i\beta$,

$$Z_{i\beta}(r,\theta) = e^{-\phi_2(\theta)} \exp\left[ i \left( \theta + \phi_1(\theta) - \frac{\ln r}{\beta} \right) \right]$$

$$m = \min\left(e^{-\phi_2(\theta)}\right) \quad M = \max\left(e^{-\phi_2(\theta)}\right), \forall \epsilon > 0$$
$$Z(\mathbb{R}^2 \backslash 0) = Z(\mathbb{D}(0,\epsilon)) = \mathbb{A}(m,M) = \{z : m \leq |z| \leq M\}$$



# Equation $Lu = 0$

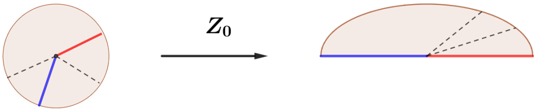
Direct consequence of order of vanishing of $L$ at 0 and definition of $Z_\mu$

- $L\left(\dfrac{\partial^{j+k}\delta}{\partial_x^j \partial_y^k}\right) = 0$ for $j+k \leq \mathrm{Re}(\lambda) - 1$;
- If $\mu = 0,\ m \in \mathbb{Z},\ \sigma m < \mathrm{Re}(\lambda) - 1,\ L(Z_0^{-m}) = 0$;
- If $\mathrm{Re}(\mu) > 0,\ m \in \mathbb{Z},\ \mathrm{Re}(\dfrac{m}{\mu}) < \mathrm{Re}(\lambda) - 1,\ L(Z_\mu^{-m}) = 0$;
- If $\mu = i\beta \in i\mathbb{R}^*,\ L(Z_{i\beta}^m) = 0,\ \forall m \in \mathbb{Z}$.

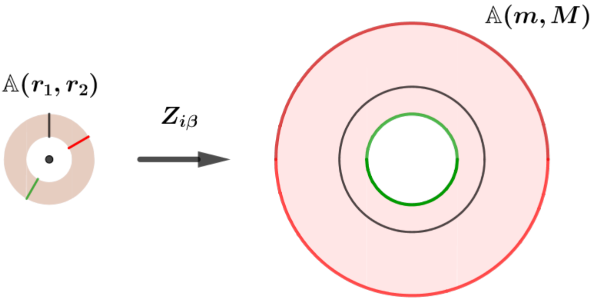
Continuous solutions.

- If $L$ satisfies condition $(\mathcal{P})$ in $\mathbb{R}^2 \backslash 0$ and $u \in C^0(\Omega)$, solves $Lu = 0$ in $\Omega$ open, then $u = H \circ Z_\mu$ with $H$ holomorphic in $Z_\mu(\Omega)$.

- If $\mu = 0$, $\Omega \ni 0$ open, $u \in C^0(\Omega)$ solves $Lu = 0$, then there is $\epsilon > 0$ such that $u \in C^\infty(\mathbb{D}(0,\epsilon) \setminus Z_0^{-1}(\mathbb{R}))$.



- If $\mu = i\beta \in i\mathbb{R}^*$, $Z_{i\beta} = \mathrm{e}^{-\phi_2(\theta)} \exp[i\,(\theta + \phi_1(\theta) - \ln r/\beta)]$. $m = \min |Z_{i\beta}|$, $M = \min |Z_{i\beta}|$. If $\Omega \supset \mathbb{A}(r_1, r_2)$ with $r_2 > r_1 \exp(2\pi/|\beta|)$, then $u \in C^0(\Omega)$, $Lu = 0$ can be written $u = H \circ Z_{i\beta}$ where $H \in C^0(\mathbb{A}(m, M))$ and holomorphic in the interior of $\mathbb{A}(m, M)$. In particular, $u$ extends as a $C^0$-solution to $\mathbb{R}^2\backslash 0$.



# Domain of extendability of solutions of $Lu = 0$

$\Omega \subset \mathbb{R}^2$ is *starlike* w.r.t. 0 if $[0,\ p] \subset \Omega,\ \forall p \in \Omega$.
Consider $L$ with $\mu \geq 0$. Define an equivalence relation on $\partial\Omega$ by

$$p \sim p' \iff \arg(Z_\mu(p)) = \arg(Z_\mu(p')); \mathrm{cl}(p) = \{p' \in \partial\Omega,\ p' \sim p\}.$$

Define $\rho :\ \partial\Omega \longrightarrow \mathbb{R}^+$; $\rho(p) = \max\limits_{p' \in \mathrm{cl}(p)} |Z_\mu(p')|$, and

$\Omega_L = \bigcup\limits_{p \in \partial\Omega} [0,\ \Lambda(p) e^{i \arg(p)})$, where $\Lambda(p) = \rho(p)^\mu e^{\mu \phi_2(\arg(p))}$ if $\mu > 0$

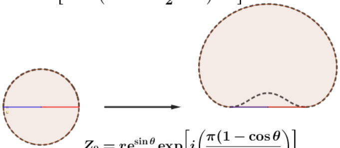
and $\Lambda(p) = \rho(p)^{1/\sigma} e^{\phi_2(\arg(p))/\sigma}$ if $\mu = 0$

- $(\Omega_L)_L = \Omega_L,\ \Omega_L = Z_\mu^{-1}(Z_\mu(\Omega))$
- If $u \in C^0(\Omega)$ solves $Lu = 0$, then there exists $\hat{u} \in C^0(\Omega_L)$, $L\hat{u} = 0$ and $\hat{u} = u$ in $\Omega$. Moreover, $\hat{u} \in C^\infty(\Omega_L \backslash 0)$ if $\mu > 0$ and $\hat{u} \in C^\infty(\Omega_L \backslash Z_0^{-1}(\mathbb{R}))$ if $\mu = 0$
- There exists $v \in C^0(\Omega_L)$ with $Lv = 0$ such that $v$ has no extension as a solution to a larger set.

# Examples

- $\mu = 0$

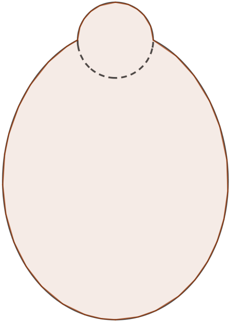
$$L = r^{\lambda-1}\Big[\partial_\theta - \Big(\cos\theta + i\frac{\pi}{2}\sin\theta\Big) r\partial_r\Big]$$



$$Z_0 = r e^{\sin\theta} \exp\left[i\left(\frac{\pi(1-\cos\theta)}{2}\right)\right]$$

$\mathbb{D}_L$ is the region enclosed by the curve $e^{(|\sin\theta| - \sin\theta)} e^{i\theta}$.

$\mathbb{D}$ **and** $\mathbb{D}_L$



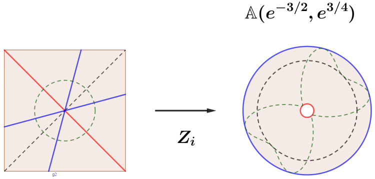
- $\mu \in \mathbb{R}^+ + i\mathbb{R}$ and $k \in \mathbb{Z}^+$

$L = r^{\lambda-1}\left[\partial_\theta - i\mu\left(1 + k\cos(k\theta)\right) r\partial_r\right]$, $Z_\mu = r^{1/\mu}\mathrm{e}^{i(\theta+\sin(k\theta))}$
$\Sigma$ consists of $2k$ rays given by $\cos\theta = -1/k$.
$L$ does not satisfy condition $(\mathcal{P})$ at any point of $\Sigma$. $Z_\mu(\mathbb{D}) = \mathbb{D}$.
If $u \in C^0(\mathbb{D})$ solves $Lu = 0$, then $u \in C^\infty(\mathbb{D}\backslash 0)$.

- $\mu = i$

$$L = r^{\lambda-1}\left[\partial_\theta - i\left(2\cos(2\theta) - 2\sin(4\theta) + i\right) r\partial_r\right],$$
$$Z_i = r^{-i}\exp\left[\sin(2\theta) + \frac{\cos(4\theta)}{2}\right] \mathrm{e}^{i\theta}$$

$\Sigma = \{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\}$ $Z_i$ is a fold along each ray of $\Sigma$. $\max|Z_i| = \mathrm{e}^{3/4}$ reached along $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
$\min|Z_i| = \mathrm{e}^{-3/2}$ reached along $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$.




If $u \in C^0(\mathbb{A}(r_1, r_2))$ with $r_2 > r_1\mathrm{e}^{2\pi}$ and $Lu = 0$, then $u$ extends as a continuous solution $\hat{u}$ to $\mathbb{R}^2\backslash 0$ and $\hat{u}$ is $C^\infty$ on the rays $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$.

# Equation $Lu = f \quad (*)$

- $f \in C^l(\mathbb{R}^2 \backslash 0)$, $\sigma$-homogeneous with $\sigma \in \mathbb{C}$ and $\mathrm{Re}(\sigma) > 0$.

• If $\mu(\lambda - \sigma - 1) \notin \mathbb{Z}$ , then $(*)$ has a solution $u \in \mathcal{D}'(\mathbb{R}^2)$ with $u \in C^{l+1}(\mathbb{R}^2 \backslash 0)$ and $(\sigma + 1 - \lambda)$-homogeneous. In particular if $\mathrm{Re}(\sigma - \lambda) > -1$, $u$ is Hölder continuous at 0.
Explicit construction of $u$: $f(r,\theta) = r^\sigma f_0(\theta)$ with $f_0 \in C^l(\mathbb{S}^1)$;
$L = r^{\lambda-1}(p(\theta)\partial_\theta - iq(\theta)r\partial_r)$, $\psi(\theta) = \int_0^\theta \frac{q}{p}\,ds$ so that $\psi(2\pi) = 2\pi\mu$.
$u(r,\theta) = r^{\sigma+\lambda-1}v(\theta)$ with

$$v(\theta) = \left[K + \int_0^\theta \frac{f_0(s)}{p(s)} \exp(-i(\sigma+1-\lambda)\psi(s))\,ds\right] \exp(i(\sigma+1-\lambda)\psi(\theta))$$

and
$$K = \frac{1}{1 - \mathrm{e}^{2\pi i\mu(\sigma+1-\lambda)}} \int_0^{2\pi} \frac{f_0(s)}{p(s)} \exp(-i(\sigma+1-\lambda)\psi(s))\,ds.$$

• If $\mu(\lambda - \sigma - 1) \in \mathbb{Z}$, equation $(*)$ we have similar conclusion provided that $f_0$ satisfies

$$\int_0^{2\pi} \frac{f_0(s)}{p(s)} \exp(-i(\sigma+1-\lambda)\psi(s))\,ds = 0.$$

- $f$ real analytic at 0.

$(\mu, \lambda)$ is *resonant* if $\exists l \in \mathbb{Z}^+,\ k \in \mathbb{Z}$ such that $\mu\lambda = \mu l + k$.
$\mathbb{J}(\mu, \lambda) = \{l \in \mathbb{Z}^+;\ \mu\lambda = \mu l + k\}$: set of resonant integers.
When $(\mu, \lambda)$ is resonant $|\mathbb{J}(\mu, \lambda)| = 1$ if $(\mu, \lambda) \notin \mathbb{Q}^+ \times \mathbb{Q}^+$ and $|\mathbb{J}(\mu, \lambda)| = \infty$ if not.
A **nonresonant** pair $(\mu, \lambda)$ satisfies Diophantine condition
$(\mathcal{DC})$: $\quad \exists C > 0,\ \forall j \in \mathbb{Z}^+,\ \left|1 - e^{2\pi i\mu(j-\lambda)}\right| \geq C^j$
- If $(\mu, \lambda)$ is nonresonant, $(\mathcal{DC})$ holds whenever $\mu \notin \mathbb{R}$ or $\mu \in \mathbb{R}^+$ and $\lambda \notin \mathbb{R}$.
- For $\mu, \lambda \in \mathbb{R}$, condition $(\mathcal{DC})$ is equivalent to

$(\mathcal{DC}')$ $\quad \exists C > 0,\ \forall j \in \mathbb{Z}^+,\ k \in \mathbb{Z},\ \ |\mu(j - \lambda) - k| \geq C^j$

- Suppose $(\mu, \lambda)$ is nonresonant and satisfies $(\mathcal{DC})$. Then for every $f$ real analytic at 0, there exists $\epsilon > 0$ and $w \in C^\infty(D(0,\epsilon)\backslash 0) \cap L^\infty(D(0,\epsilon))$ such that $u = \dfrac{w}{r^{\lambda-1}}$ is a distribution solution of $Lu = f$.

$f(x,y) = \sum_{j\geq 0} P_j(x,y)$, with $P_j$ homogeneous polynomial of degree $j$.
$(\mu, \lambda)$ nonresonant $\Longrightarrow$ $Lu = P_j$ has solution $r^{j+1-\lambda} v_j(\theta)$.
$(\mu, \lambda)$ satisfies $(\mathcal{DC}') \Longrightarrow$
$w = \sum_j r^j v_j(\theta) \in C^\infty(D(0,\epsilon)\backslash 0) \cap L^\infty(D(0,\epsilon))$ and $L(w/r^{\lambda-1}) = f$.

- Suppose $(\mu, \lambda)$ is resonant. A real analytic function $f = \sum_j r^j f_j(\theta)$ is $(\mu, \lambda)$-*compatible* if

$$\int_0^{2\pi} \frac{f_j(s)}{p(s)} \mathrm{e}^{-i(j-1-\lambda)\psi(s)} ds = 0 \qquad \forall j \in \mathbb{J}(\mu, \lambda).$$

Note: If $\mu \notin \mathbb{Q}$, there is only one compatibility condition and if $\mu \in \mathbb{Q}$ there are infinitely many conditions.
Suppose $(\mu, \lambda)$ is resonant and $f$ is $(\mu, \lambda)$-*compatible*, then $Lu = f$ has a solution as above.

# Integral operator and Hölder continuous solutions

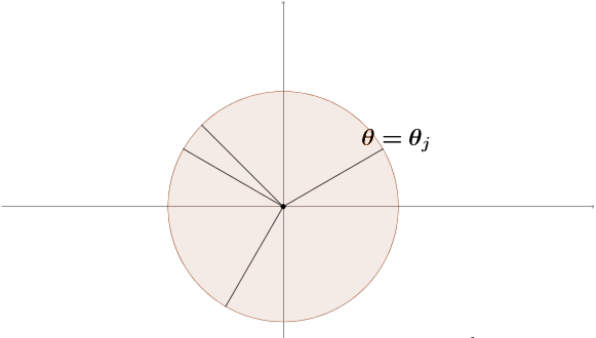
(join work with C. Campana and P. Dattori)

Class of vector fields $L$ such that $(\imath)$ $\lambda\mu = 1$; $(\imath\imath)$ $\lambda \in \mathbb{R}^+$; $(\imath\imath\imath)$ satisfies condition $(\mathcal{P})$ in $\mathbb{R}^2\backslash 0$; and $(\imath v)$ $L$ is of finite type.

There exist coordinates $(r, \theta)$ such that $Z = r^\lambda e^{i(\theta+\phi(\theta))}$ is a first integral with $\phi \in C^\infty(\mathbb{S}^1, \mathbb{R})$;

$1 + \phi'(\theta) \geq 0$ for all $\theta$; and

Taylor series of $\phi''$ not identically 0 at for every $\theta_j \in \Sigma_0$.




$k_j$ smallest (odd) integer such that $\phi^{k_j}(\theta_j) \neq 0$

$k^\bullet = \max k_j,\ \tau = \dfrac{k^\bullet - 1}{k^\bullet}$. Let $\Omega$ open in $\mathbb{R}^+ \times \mathbb{S}^1$.

For $0 < \lambda < 1$, let $\mathcal{F} = L^p(\Omega)$ with $p > \max\{k^\bullet + 1, 1/(1-\lambda)\}$.

For $\lambda \geq 1$, let $\delta > 0$ with $1 - \dfrac{1}{\lambda} < \delta < 1 - \dfrac{1}{\lambda(2-\tau)}$,

$\mathcal{F} = r^{\lambda\delta} L^p(\Omega)$ with $p > 1/[1 - \lambda(1-\delta)]$.

$$\|f\|_{\mathcal{F}} = \begin{cases} \|f\|_p & \text{when } \mathcal{F} = L^p(\Omega) \\ \|f_0\|_p & \text{when } \mathcal{F} = r^{\lambda\delta} L^p(\Omega) \text{ and } f = r^{\lambda\delta} f_0 \end{cases}$$

For $f \in \mathcal{F}$, define

$$T_Z f(r,\theta) = \frac{1}{2\pi i} \int_\Omega \frac{f(\xi,\eta)}{Z(\xi,\eta) - Z(r,\theta)}\, d\xi d\eta\,.$$

- $\exists M > 0;\ \ |T_Z f(p)| \leq M \|f\|_{\mathcal{F}} \ \ \forall f \in \mathcal{F},\ p \in \Omega$
- $\exists C > 0,\ \beta > 0$ such that $\forall f \in \mathcal{F},\ p_1, p_2 \in \Omega$

$$|T_Z f(p_1) - T_Z f(p_2)| \leq C\, |Z(p_1) - Z(p_2)|^{\beta}\,.$$

- $L(T_Z f) = f.$

Obrigado