Fall 2022 - Real Analysis
Homework about the
Cantor Set and Cantor-Lebesgue function

The aim of these exercises is to give an alternative view, based on the ternary and binary expansions of real numbers, of the Cantor set and of the Cantor-Lebesgue function.

Let $p \in \mathbb{N}$ with $p > 1$. We have seen in a previous exercise that any $x \in [0,\ 1]$ can be written as $x = \sum_{j=1}^{\infty} \frac{a_j}{p^j}$ with $a_j \in \{0, 1, \cdots, p-1\}$. This is a representation of x in base p. We will use the following notation.

$$x = [0.a_1 a_2 \cdots a_n \cdots]_p.$$

We have also seen that such a representation is unique except when $x = \frac{q}{p^N}$ for some natural number $q < p^N$. In this case x has two representations

$$x = [0.a_1 a_2 \cdots a_m 1000 \cdots]_p = [0.a_1 a_2 \cdots a_m 0(p-1)(p-1)(p-1)\cdots]_p\,.$$

From now on when $x = \frac{q}{p^N}$ we will use only the second representation:

$$x = [0.a_1 a_2 \cdots a_m \mathbf{0}(\mathbf{p-1})(\mathbf{p-1})(\mathbf{p-1})\cdots]_p$$

and use the notation $x = [0.a_1 a_2 \cdots a_m 0\overline{(p-1)}]_p$ to indicate that the $(p-1)$ repeats infinitely in the representation. With this understanding any $x \in [0,\ 1]$ has a unique expansion in base p.

(1) Let $I = (a_1,\ b_1)$ the open third middle interval removed from $C_0 = [0,\ 1]$ in the construction of the Cantor set C. Verify that
$$a_1 = [0.0\overline{2}\,]_3 \quad \text{and} \quad b_1 = [0.2\,]_3\,.$$

(2) Let $I = (a_2,\ b_2)$ the one of the two open third middle intervals removed from $C_1 = C_0 \backslash (a_1,\ b_1)$. Verify that
$$a_2 = [0.00\overline{2}\,]_3 \ \text{ and } \ b_2 = [0.02\,]_3 \quad \text{or} \quad a_2 = [0.20\overline{2}\,]_3 \ \text{ and } \ b_2 = [0.22\,]_3\,.$$

(3) In general if $I = (a_n,\ b_n)$ is one of the 2^n open third middle intervals removed from the components of C_{n-1}, show that the expansions in base 3 of the endpoints a_n and b_n have the form
$$a_n = [0.\alpha_1\alpha_2 \cdots \alpha_{n-1}0\overline{2}\,]_3 \quad \text{and} \quad b_1 = [0.\alpha_1\alpha_2 \cdots \alpha_{n-1}2\,]_3\,,$$
with $\alpha_j = 0$ or 2 for $j = 1, \cdots, n-1$.

(4) Show that if x is in one of the removed open intervals (i.e. $x \in (a_n,\ b_n)$), then its ternary expansion contains the digit 1. That is, $x \in (a_n,\ b_n)$ if and only if
$$x = [0.x_1x_2x_3\cdots]_3 \quad \text{and} \quad x_j = 1 \ \text{ for some } j \in \mathbb{N}\,.$$

(5) Deduce that the Cantor set C consists of the real numbers $x \in [0,\ 1]$ such that
$$x = [0.x_1x_2x_3\cdots]_3 \quad \text{with} \quad x_j \in \{0, 2\}, \quad \forall j \in \mathbb{N}\,.$$

(6) Consider the function $\phi_0 : C \longrightarrow [0,\ 1]$ defined as follows: For $x \in C$ with $x = [0.x_1x_2x_3\cdots]_3$ define $\phi_0(x)$ by
$$\phi_0(x) = \left[0.\frac{x_1}{2}\frac{x_2}{2}\frac{x_3}{2}\cdots\right]_2$$
Verify that the function ϕ_0 is well defined.

(7) Prove that ϕ_0 is continuous.

(8) Prove that ϕ_0 extends as a function $\phi :\ [0,\ 1] \longrightarrow [0,\ 1]$ so that ϕ is continuous and is constant on each removed middle third interval $(a_n,\ b_n)$.

(9) Verify that this function coincides with the Cantor-Lebesgue function defined in the Lecture Notes as the limit of the f_n's.