Spring 2022 – MAP4401
Homework I

This homework is due by **Tuesday February 15**. (I will not accept late returns). Clarity of exposition is expected and no credits will be given for sloppy work. I trust that your work will be individual and you are free to consult notes and books. You have to turn in the following 9 problems.

Problem 1. (10pts) Write in formal style the BVP for the temperature in a ball of radius R made up of material with diffusivity K and such that the upper hemisphere is kept at constant temperature T_1 and the lower hemisphere is kept at a constant temperature T_2. Use spherical coordinates in the sphere.
Let $u(\rho, \theta, \phi, t)$ be the temperature at the point $((\rho, \theta, \phi)$ at time t, then u satisfies the BVP

$$\begin{cases} u_t = k\Delta u & \text{for } \ 0 \le \rho < R, \ \ 0 < \theta < 2\pi, \ \ 0 < \phi < \pi, \ \ t > 0 \\ u(R, \theta, \phi, t) = T_1 & \text{for } \ 0 < \theta < 2\pi, \ \ 0 < \phi < \dfrac{\pi}{2}, \ \ t > 0 \\ u(R, \theta, \phi, t) = T_2 & \text{for } \ 0 < \theta < 2\pi, \ \ \dfrac{\pi}{2} < \phi < \pi, \ \ t > 0 \end{cases}$$

where Δ is the Laplace operator given in spherical coordinates by

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2 \sin^2 \phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\cot \phi}{\rho^2}\frac{\partial}{\partial \phi}$$

Problem 2. (10 pts) Write in formal style the BVP for the temperature in a circular plate (disk) of radius R made up of material with diffusivity K and such that one half of the boundary (say the upper semicircle) is kept at constant temperature T_0, and the other half of the boundary (lower semicircle)is insulated. Assume that the disk is laterally insulated (so there is no heat flow in the direction perpendicular to the disk). Use polar coordinates in the disk.
Let $u(r, \theta, t)$ be the temperature at the point $((r, \theta)$ at time t, then u satisfies the BVP

$$\begin{cases} u_t = k\Delta u & \text{for } \ 0 \le r < R, \ \ 0 < \theta < 2\pi, \ \ t > 0 \\ u(R, \theta, t) = T_0 & \text{for } \ 0 < \theta < \pi, \ \ t > 0 \\ u_r(R, \theta, t) = 0 & \text{for } \ \pi < \theta < 2\pi, \ \ t > 0 \end{cases}$$

where Δ is the Laplace operator given in polar coordinates by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Problem 3. (10 pts) Write in formal style the BVP for the vibrations of a string subject to the following. The string has length L (in centimeters); its end points are held fixed; there is a damping with coefficient a (resistance to the motion); assume that initially the string is pulled by 1 cm at the point $x = L/4$ and then released from rest; (for simplicity assume that the speed of the wave c is normalized to be 1)
Let $u(x, t)$ be the vertical displacement at time t of the point x on the string. Then u satisfies the BVP

$$\begin{cases} u_{tt} + au_t = u_{xx} & \text{for } \ 0 < x < L, \ \ t > 0 \\ u(0, t) = 0, \ \ u(L, t) = 0 & \text{for } \ t > 0 \\ u(x, 0) = f(x), \ \ u_t(x, 0) = 0 & \text{for } \ 0 < x < L \end{cases}$$

where

$$f(x) = \begin{cases} \dfrac{4x}{L} & \text{for } 0 < x < \dfrac{L}{4} \\ \dfrac{-4(x-L)}{3L} & \text{for } \dfrac{L}{4} < x < L \end{cases}$$

Problem 4. (10 pts) Write in formal style the BVP for the vibrations of a drum head (membrane) subject to the following. The drum head is a circular membrane of radius R with a fixed boundary. Assume that the motion of the drum (which was at rest) is started by hitting it at its center with a circular object of radius $R/10$ which has a downward velocity of 2cm/sec. (Assume the speed of the wave is normalized to be 1).

Let $u(r, \theta, t)$ be the vertical displacement at time t of the point $((r, \theta)$ on the drum-head. Then u satisfies the BVP

$$\begin{cases} u_t = u_{rr} + \dfrac{1}{r} u_r + \dfrac{1}{r^2} u_{\theta\theta} & \text{for } r < R, \quad 0 \le \theta \le 2\pi, \quad t > 0 \\ u(r, \theta, 0) = 0, \quad u_r(r, \theta, 0) = g(r, \theta) & \text{for } r < R, \quad 0 \le \theta \le 2\pi, \end{cases}$$

where where

$$g(r, \theta) = \begin{cases} -2 & \text{for } r < \dfrac{R}{10} \\ 0 & \text{for } \dfrac{R}{10} < r < R \end{cases}$$

Problem 5. (20 pts) Use the method of separation of variables to solve the BVP

$$\begin{array}{ll} u_t(x,t) = 2u_{xx}(x,t) & 0 < x < 5, \quad t > 0 \\ u_x(0,t) = u_x(5,t) = 0 & t > 0 \\ u(x,0) = 100 - 7\cos\dfrac{3\pi x}{5} & 0 < x < 5 \end{array}$$

The homogeneous and nonhomogeneous parts of the BVP are:

$$(HP): \quad \begin{array}{l} u_t(x,t) = 2u_{xx}(x,t) \\ u_x(0,t) = u_x(5,t) = 0 \end{array}, \qquad (NHP) \quad u(x,0) = 100 - 7\cos\frac{3\pi x}{5}$$

If $u(x,t) = X(x)T(t)$ is a non trivial solution of (HP), then the functions X and T solve the ODE problems

$$\begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(5) = 0 \end{array}, \qquad T'(t) + 2\lambda T(t) = 0$$

where λ is the separation constant.

Now we find the eigenvalus and eigenfunctions of the X-problem. The characteristic equation of the X-ode is $m^2 + \lambda = 0$. Consider 3 cases

- $\lambda < 0$. Set $\lambda = -\nu^2$ (with $\nu > 0$). In this case the general solution is $X(x) = Ae^{\nu x} + Be^{-\nu x}$. We have $X'(x) = \nu Ae^{\nu x} - \nu Be^{-\nu x}$. In order for such function to satisfy the endpoints conditions $X'(0) = 0$ and $X'(5) = 0$, we need to have

  $$\nu(A - B) = 0 \qquad \nu(Ae^{5\nu} - Be^{-5\nu}) = 0 .$$

  Since $\nu > 0$, the only solution of this system is $A = B = 0$. This implies $X = 0$ and so $\lambda < 0$ cannot be an eigenvalue.

- $\lambda = 0$. The general solution in this case is $X(x) = A + Bx$. We have $X'(x) = B$. In order for such function to satisfy the endpoints conditions, we need $B = 0$ and arbitrary. This means that $\lambda = 0$ is an eigenvalue with eigenfunction $X_0(x) = 1$.
- $\lambda > 0$. Set $\lambda = \nu^2$ (with $\nu > 0$). In this case the general solution is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. We have $X'(x) = -\nu A\sin(\nu x) + \nu B\cos(\nu x)$. In order for such function to satisfy the endpoints conditions $X'(0) = 0$ and $X'(5) = 0$, we need to have
$$\nu B = 0 \qquad -\nu A\sin(5\nu) + \nu B\cos(5\nu) = 0\,.$$
Since $\nu > 0$, then $B = 0$ and then $A\sin(5\nu) = 0$. We get a nontrivial solution if $\sin(5\nu) = 0$. That is $\nu = \dfrac{k\pi}{5}$ with $k \in \mathbb{Z}^+$. The eigenvalues are $\lambda_k = \nu_k^2$ with $\nu_k = \dfrac{k\pi}{5}$ and the corresponding eigenfunctions are $X_k(x) = \cos(\nu_k x)$

The corresponding solutions of the T-problem: For $\lambda_0 = 0$, an independent solution is $T_0(t) = 1$. For $\lambda_k = \nu_k^2$, an independent solution is $T_k(t) = \mathrm{e}^{-2\nu_k^2 t}$.

Solutions with separated variables of (HP) are
$$u_0(x,t) = 1 \quad \text{and} \quad u_k(x,t) = \mathrm{e}^{-2\nu_k^2 t}\cos(\nu_k x), \quad \text{with} \quad \nu_k = \frac{k\pi}{5}.$$
The principle of superposition gives a more general solution:
$$u(x,t) = A_0 + \sum_{k=1}^{N} A_k \mathrm{e}^{-2\nu_k^2 t}\cos(\nu_k x)$$
To find the coefficients, we use the nonhomogeneous condition:
$$u(x,0) = A_0 + \sum_{k=1}^{N} A_k \cos\frac{k\pi x}{5} = 100 - 7\cos\frac{3\pi x}{5}\,.$$
An identification of the coefficients gives
$$A_k = 0 \;\; \text{for } k \neq 0 \;\; \text{and } k \neq 3, \quad A_0 = 100, \quad A_3 = -7$$
the solution of the BVP is
$$u(x,t) = 100 - 7\mathrm{e}^{-18\pi^2 t/25}\cos\frac{3\pi x}{5}\,.$$

Problem 6. (20 pts) Use the method of separation of variables to solve the BVP
$$\begin{array}{ll} u_{tt}(x,t) + (0.2)u_t(x,t) + (0.01)u = (0.01)u_{xx}(x,t) & 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = \sin 3x - \dfrac{1}{5}\sin 7x & 0 < x < \pi \end{array}$$
The homogeneous and nonhomogeneous parts of the BVP are:
$$(HP): \left\{ \begin{array}{l} u_{tt}(x,t) + (0.2)u_t(x,t) + (0.01)u = (0.01)u_{xx}(x,t) \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 0 \end{array} \right. ,$$
$$(NHP) \quad u_t(x,0) = \sin 3x - \frac{1}{5}\sin 7x$$

If $u(x,t) = X(x)T(t)$ is a non trivial solution of (HP), then the functions X and T solve the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}, \quad \begin{cases} T''(t) + 0.2T'(t) + 0.01(1+\lambda)T(t) = 0 \\ T(0) = 0 \end{cases}$$

where λ is the separation constant.

The eigenvalues and eigenfunctions of the SL-problem (X-problem) are

$$\lambda_n = n^2, \quad \text{and} \quad X_n(x) = \sin(nx) \quad n \in \mathbb{Z}^+.$$

The corresponding T-ODE has characteristic roots $m = \dfrac{-1}{10} \pm i\dfrac{n}{10}$ and independent solutions $\mathrm{e}^{-0.1t}\cos(0.1nt)$ and $\mathrm{e}^{-0.1t}\sin(0.1nt)$. Since in addition, we need $T(0) = 0$, then the T-problem has one independent solution: $\mathrm{e}^{-0.1t}\sin(0.1nt)$.

Solutions with separated variables of (HP) are $\mathrm{e}^{-0.1t}\sin(0.1nt)\, sin(nx)$. The principle of superposition gives a more general solution:

$$u(x,t) = \sum_{n=1}^{N} C_n \mathrm{e}^{-0.1t}\sin(0.1nt)\, sin(nx)\,.$$

To find the coefficients, we use the nonhomogeneous condition. We have

$$u_t(x,t) = \sum_{n=1}^{N} C_n \mathrm{e}^{-0.1t}\left[-0.1\sin(0.1nt) + 0.1n\cos(0.1nt)\right]\sin(nx)$$

Hence

$$u_t(x,0) = \sum_{n=1}^{N} 0.1n\, C_n \sin(nx) = \sin 3x - \frac{1}{5}\sin 7x$$

An identification of the coefficients gives

$$C_n = 0 \ \text{ for } n \neq 3 \ \text{ and } n \neq 7, \quad C_3 = \frac{10}{3}, \quad C_7 = \frac{-2}{7}.$$

the solution of the BVP is

$$u(x,t) = \mathrm{e}^{-t/10}\left[\frac{10}{3}\sin(0.3t)\sin(3x) - \frac{2}{7}\sin(0.7t)\sin(7x)\right].$$

Problem 7. (20 pts) Use the method of separation of variables to solve the BVP

$$\begin{array}{ll} \Delta u(x,y) = 0 & 0 < x < \pi, \quad 0 < y < \pi \\ u_x(0,y) = u_x(\pi,y) = 0 & 0 < y < \pi \\ u(x,0) = 0 & 0 < x < \pi \\ u(x,\pi) = \cos 3x - \dfrac{1}{5}\cos(9x) & 0 < x < \pi \end{array}$$

The homogeneous and nonhomogeneous parts of the BVP are:

$$(HP): \begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 \\ u_x(0,y) = 0, \quad u_x(\pi,y) = 0 \\ u(x,0) = 0 \end{cases},$$

$$(NHP) \quad u(x,\pi) = \cos 3x - \frac{1}{5}\cos(9x)$$

If $u(x,t) = X(x)Y(y)$ is a non trivial solution of (HP), then the functions X and Y solve the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases} , \quad \begin{cases} Y''(y) - \lambda Y(y) = 0 \\ Y(0) = 0 \end{cases}$$

where λ is the separation constant.

The eigenvalues and eigenfunctions of the SL-problem (X-problem) are

$$\lambda_n = n^2, \quad \text{and} \quad X_n(x) = \cos(nx) \quad n = 0,\ 1,\ 2,\ 3,\ \cdots$$

The corresponding Y-ODE has characteristic roots $m = \pm n$. For $n = 0$, the general solution is $Y(y) = A + By$ and for $Y(0) = 0$, we need $A = 0$. For $n > 0$, the general solution is $Y(y) = Ae^{ny} + Be^{-ny}$. In order to have $Y(0) = 0$, we need $A + B = 0$ and so $Y(y) = A(e^{ny} - e^{-ny}) = 2A\sinh(ny)$.

Solutions with separated variables of (HP) are therefore

$$y \ \text{ for } \ n = 0 \quad \text{and} \quad \sinh(ny)\cos(nx) \ \text{ for } n = 1,\ 2,\ \cdots$$

The principle of superposition gives a more general solution:

$$u(x,y) = A_0 y + \sum_{n=1}^{N} A_n \sinh(ny)\cos(nx).$$

To find the coefficients, we use the nonhomogeneous condition. We have

$$u(x,\pi) = A_0\pi + \sum_{n=1}^{N} A_n \sinh(n\pi)\cos(nx) = \cos(3x) - \frac{1}{5}\cos(9x)$$

An identification of the coefficients gives

$$A_n = 0 \ \text{ for } n \neq 3 \ \text{ and } n \neq 9, \quad A_3\sinh(3\pi) = 1, \quad A_9\sinh(9\pi) = \frac{-1}{5}.$$

the solution of the BVP is

$$u(x,y) = \frac{\sinh(3y)}{\sinh(3\pi)}\cos(3x) - \frac{\sinh(9y)}{5\sinh(9\pi)}\cos(9x)\,.$$

Problem 8. (20 pts) Use the method of separation of variables to solve the BVP given in polar coordinates by

$$\begin{array}{ll} \Delta u(r,\theta) = 0 & 0 < r < 2, \quad 0 \le \theta \le 2\pi \\ u_r(2,\theta) = \sin\theta + \cos(5\theta) & 0 \le \theta \le 2\pi \end{array}$$

Since we are using polar coordinates the function u and its continuous derivatives satisfies $u(r,0) = u(r,2\pi)$, $u_\theta(r,0) = u_\theta(r,2\pi)$. The homogeneous and nonhomogeneous parts of the BVP are:

$$(HP): \quad \begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0 \\ u(r,0) = u(r,2\pi) \quad u_\theta(r,0) = u_\theta(r,2\pi) \end{cases} ,$$

$$(NHP) \quad u_r(2,\theta) = \sin\theta + \cos(5\theta)$$

If $u(r,\theta) = R(r)\Theta(\theta))$ is a non trivial solution of (HP), then the functions R and Θ solve the ODE problems

$$\begin{cases} \Theta''(x) + \lambda\Theta(\theta) = 0 \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}, \quad r^2R''(r) + rR'(r) + \lambda R(r) = 0\,.$$

where λ is the separation constant. The eigenvalues and eigenfunctions of the periodic Θ-problem are

$\lambda_0 = 0,\ \Theta_0(\theta) = 1, \quad \lambda_n = n^2,\ \Theta_{n,1}(\theta) = \cos(n\theta)$ and $\Theta_{n,2}(\theta) = \sin(n\theta),\ n \in \mathbb{Z}^+.$

The corresponding independent solutions of the R-ode (Cauchy-Euler) are

$R_{0,1}(r) = 1,\ R_{0,2}(r) = \ln r$ (for $\lambda_0 = 0$) and $R_{n,1}(r) = r^n,\ R_{n,2}(r) = \dfrac{1}{r^n}$ (for $\lambda_n = n^2 > 0$)

Since $R_{0,2}$ and $R_{n,2}$ are not bounded as $r \longrightarrow 0^+$, then the bounded solutions with separated variables of the (HP) are

$$1, \quad r^n\cos(n\theta), \quad \text{and} \quad r^n\sin(n\theta) \quad \text{with } n \in \mathbb{Z}^+.$$

The principle of superposition gives a more general solution:

$$u(r,\theta) = A_0 + \sum_{n=1}^{N} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right].$$

We (NHP) to find the coefficients. We have

$$u_r(r,\theta) = \sum_{n=1}^{N} nr^{n-1}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right].$$

so that

$$u_r(2,\theta) = \sum_{n=1}^{N} n2^{n-1}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right] = \sin(\theta) + \cos(5\theta)$$

Hence all coefficients are 0 except $B_1 = 1$ and $A_1 = \dfrac{1}{80}$ and A_0 arbitrary The solution of the BVP is

$$u(r,\theta) = A_0 + \sin\theta + \frac{r^5}{80}\cos(5\theta)$$

Problem 9. (20 pts) Use the method of separation of variables to solve the BVP

$$\begin{array}{ll} u_t(x,y,t) = \Delta u(x,y,t) & 0 < x < \pi,\quad 0 < y < 2\pi,\quad t > 0 \\ u(0,y,t) = u(\pi,y,t) = 0 & 0 < y < 2\pi,\quad t > 0 \\ u_y(x,0,t) = u_y(x,2\pi,t) = 0 & 0 < x < \pi,\quad t > 0 \\ u(x,y,0) = \sin x + \cos\dfrac{y}{2}\sin 2x & 0 < x < \pi,\quad 0 < y < 2\pi \end{array}$$

The homogeneous and nonhomogeneous parts of the BVP are:

$$(HP): \begin{cases} u_t = u_{xx}(x,y) + u_{yy}(x,y) \\ u(0,y,t) = u(\pi,y,t) = 0 \\ u_y(x,0,t) = u_y(x,2\pi,t) = 0 \end{cases},$$

$$(NHP) \quad u(x,y,0) = \sin x + \cos\frac{y}{2}\sin 2x$$

If $u(x,y,t) = X(x)Y(y)T(t)$ is a non trivial solution of (HP), then the functions X, Y and T solve the ODE problems

$$\begin{cases} X''(x) + \alpha X(x) = 0 \\ X(0) = 0, \quad X(\pi) = 0 \end{cases}, \quad \begin{cases} Y''(y) + \beta Y(y) = 0 \\ Y'(0) = 0, \quad Y'(2\pi) = 0 \end{cases}$$

and

$$T'(t) + \lambda T(t) = 0$$

where $\alpha,\ \beta,\ \lambda$ is the separation constants with $\lambda = \alpha + \beta$.

The eigenvalues and eigenfunctions of the X-problem and of the Y-problem are

$$\alpha_n = n^2, \quad X_n(x) = \sin(nx) \quad n = 1,\ 2,\ 3,\ \cdots$$

$$\beta_m = \left(\frac{m}{2}\right)^2, \quad Y_m(y) = \cos\frac{my}{2}$$

For these values of α_n and β_m, we have $\lambda_{nm} = \alpha_n^2 + \beta_m^2$ and an independent solution of the corresponding T-equation is

$$\mathrm{e}^{-\lambda_{nm}t} = \exp\left[-\left(n^2 + \frac{m^2}{4}\right)t\right]$$

Solutions with separated variables of (HP) are therefore

$$\mathrm{e}^{-\lambda_{nm}t}\sin(\alpha_n x)\cos(\beta_m y) \text{ for } n = 1,\ 2,\ \cdots \text{ and } m = 0,\ 1,\ 2,\ cdots$$

The principle of superposition gives a more general solution:

$$u(x,y,t) = \sum_{n=1}^{N}\sum_{m=1}^{M} C_{n,m}\mathrm{e}^{-\lambda_{nm}t}\sin(\alpha_n x)\cos(\beta_m y).$$

To find the coefficients, we use the nonhomogeneous condition. We have

$$u(x,y,0) = \sum_{n=1}^{N}\sum_{m=1}^{M} C_{n,m}\sin(nx)\cos\frac{my}{2} = \sin x + \cos\frac{y}{2}\sin 2x$$

An identification of the coefficients gives $C_{nm} = 0$ for all (n,m) except for $C_{1,0} = 1$ and $C_{2,1} = 1$. The solution of the BVP is

$$u(x,y,t) = \mathrm{e}^{-\lambda_{1,0}t}\sin x + \mathrm{e}^{-\lambda_{2,1}t}\sin(2x)\cos\frac{y}{2} = \mathrm{e}^{-t}\sin x + \mathrm{e}^{-17t/4}\sin(2x)\cos\frac{y}{2}\,.$$