Lecture 1
Real Analysis MAA 6616

Review of basic properties of $\mathbb{R}$

- Notation
- Least upper bound and greatest lower bound
- Density in $\mathbb{R}$
- Countable and uncountable sets

Notation

- $\mathbb{N} = \{1, 2\,3 \cdots\}$: The set of natural numbers
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$: The set of integers
- $\mathbb{Q} = \left\{ x = \frac{p}{q};\ \ p, q \in \mathbb{Z},\ q \neq 0 \right\}$: The set of rational numbers
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R};\ \ x > 0\}$: The set of positive real numbers.
- If E and F are nonempty sets, the cartesian product $E \times F$ denotes the set of all elements (x, y) such that $x \in E$ and $y \in F$.

Supremum and infimum

A set $E \subset \mathbb{R}$ is bounded above if there exists $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in E$ such a number b is called an upper bound for E.

Completeness Axiom: *Let $E \subset \mathbb{R}$ be bounded above. Then the set of all upper bounds has a smallest element*

If $U = \{b \in \mathbb{R}; \ \ b \text{ upper bound for } E\}$, then there is $b_0 \in U$ such that $b_0 \leq b$ for all $b \in U$.

b_0 is called the least upper bound for E or the supremum of E.

$$b_0 = \mathrm{l.u.b}(E) = \sup(E)$$

Example. $E = \{x \in \mathbb{R}; \ \ x^2 < 2\}$ is bounded above and $\sup(E) = \sqrt{2}$. Note that in this example $\sup(E) \notin E$

A set $E \subset \mathbb{R}$ is bounded below if there exists $a \in \mathbb{R}$ such that $x \geq b$ for all $x \in E$ such a number a is called a lower bound for E.

Theorem

Let $E \subset \mathbb{R}$ be bounded below. Then E has a largest lower bound. If $U = \{a \in \mathbb{R};\ \ a \text{ lower bound for } E\}$, then there is $a_0 \in U$ such that $a_0 \leq a$ for all $a \in U$.

This theorem is consequence of the completeness axiom. Indeed, if E is bounded below then the set $E' = \{x \in \mathbb{R};\ -x \in E\}$ is bounded above and so has l.u.b b_0. Then using the completeness axiom, we can show that a_0 is largest lower bound for E.

a_0 is called the greatest lower bound for E or the infimum of E.

$$a_0 = \text{g.l.b.}(E) = \inf(E)$$

Density in $\mathbb{R}$

$\triangle$ **inequality:** For every $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$

Archimedean property: For every $a, b \in \mathbb{R}^+$ there exists $n \in \mathbb{N}$ such that $na > b$.
In particular, for any given $\epsilon > 0$ (no matter how small) there exists $n \in \mathbb{N}$ such that $n\epsilon > 1$.

A subset E of $\mathbb{R}$ is said to be dense in $\mathbb{R}$ if for every $x, y \in \mathbb{R}$ with $x < y$ the interval (x, y) contains elements of E $(E \cap (x, y) \neq \emptyset)$.

Theorem

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof.

Let $a, b \in \mathbb{R}$. Suppose $0 < a < b$. then $b - a > 0$.It follows from the Archimedean property that there exists $n \in \mathbb{N}$ such that $n(b - a) > 1$ or equivalently $\frac{1}{n} < b - a$.

Let $S_n = \left\{ m \in \mathbb{N} \text{ such that } \frac{m}{n} > b \right\}$ by the Archimedean property S_n is a nonempty subset of $\mathbb{N}$. Therefore S_n has a smallest element p. ($p \in S_n$ and $(p - 1) \notin S_N$).

We have then

$$\frac{p-1}{n} \leq b; \ \frac{p}{n} > b; \text{ and } \frac{1}{n} < b - a$$

It follows from these inequalities that the rational number $r = \frac{p-1}{n} \in \mathbb{Q}$ satisfies $a < r < b$.

If $a \leq 0$ and $b > 0$, then by the Archimedean property there is $n \in \mathbb{N}$ so that the rational number $r = \frac{1}{n}$ satisfies $a \leq 0 < r < b$.

If $a < b < 0$, then $0 < -b < -a$. The above argument shows that there exists $r \in \mathbb{Q} \cap (-b, \ -a)$. Therefore $-r \in \mathbb{Q}$ satisfies $a < -r < -b$.

□

Countable and uncountable sets

Two sets E and F are said to be equipotent if there exists a bijection $f : E \longrightarrow F$ (f is injective and surjective).

A set E is said to be finite if either $E = \emptyset$ or there exists $n \in \mathbb{N}$ such that E is equipotent to the set $\{1, \cdots n\}$ (n is number of elements in E).

A set E is said to be countably infinite if E is equipotent to $\mathbb{N}$.

A set E is said to be countable if E is either finite or countably infinite. A set that is not countable is said to be uncountable.

Theorem

A subset of a countable set is countable. In particular if $E \subset \mathbb{N}$, then E is countable

Proof.

Case 1. F is finite: Let n be the cardinality of F so that there exists a bijection $f : \{1, \cdots n\} \longrightarrow F$. Since $E \subset F$, then there exists a first integer j_1 such that $1 \leq j_1 \leq n$ and $f(j_1) \in E$. If $E = \{f(j_1)\}$ then the function $h : \{1\} \longrightarrow E$ given by $h(1) = f(j_1)$ is a bijection and E has cardinality 1. If $E \backslash \{h(1)\} \neq \emptyset$, then let j_2 be the smallest integer such that $f(j_2) \in E \backslash \{h(1)\}$ and define $h(2) = f(j_2)$. If $E \backslash \{h(1), h(2)\} = \emptyset$, then E is finite with cardinality 2. This selection process terminates after p steps with $p \leq n$ so that E is finite with p elements.

Case 2. F is countably infinite: Let $f : \mathbb{N} \longrightarrow F$ be a bijection. Let $j_1 \in \mathbb{N}$ be the first integer such that $f(j_1) \in E$. Set $h(1) = f(j_1)$.Repeat the selection for $h(2), h(3), \cdots$ as in the previous case. If this selection process terminates after N steps, then (by construction) E has N elements. If the selection process does not terminate, then the function h is defined on $\mathbb{N}$ and is injective by construction. To complete the proof, we need to show that h is surjective. First note that it can be verified by induction that for every $k \in \mathbb{N}$, we have $k \leq j_k$ with $h(k) = f(j_k)$. Now let $x \in E$. There exists $m \in \mathbb{N}$ such that $x = f(m)$ and therefore $x \in \{h(1), \cdots h(m)\}$. $\square$

Theorem

Let E and F be countable sets. Then $E \times F$ is countable. More generally if $E_1, \cdots E_n$ are countable sets, then their cartesian product $E_1 \times \cdots \times E_n$ is countable.

Proof.

Since E and F are countable then they are equipotent to subsets of $\mathbb{N}$. hence there exist injective functions $f: E \longrightarrow \mathbb{N}$ and $g: F \longrightarrow \mathbb{N}$. To prove that $E \times F$, it is enough to prove that it is equipotent to a subset of $\mathbb{N}$. For this it is enough to construct an injective function $h: E \times F \longrightarrow \mathbb{N}$. For $(x, y) \in E \times F$, define $h(x, y)$ by $h(x, y) = (f(x) + g(y))^2 + g(y)$. The function h is injective. Indeed, assume that $h(a, b) = h(x, y)$. If $g(b) = g(y)$, then $b = y$ (g injective) and it follows from the definition of h that $(f(a) + g(b))^2 + g(b) = (f(x) + g(y))^2 + g(y)$ and so $f(a) = f(x)$ (all these functions are $\mathbb{N}$-valued) consequently $a = x$. In this case we have $(a, b) = (x, y)$. Now we claim that $g(b) \neq g(y)$ leads to an absurdity. If $g(b) \neq g(y)$, then it follows from $h(a, b) = h(x, y)$ that

$$
\begin{aligned}
&(f(a) + g(b))^2 - (f(x) + g(y))^2 = g(y) - g(b) \\
&|f(a) + g(b) + f(x) + g(y)|\,|f(a) + g(b) - f(x) - g(y)| = |g(y) - g(b)|
\end{aligned}
$$

This means that $f(a) + g(b) + f(x) + g(y) > |g(y) - g(b)|$ and divides $|g(y) - g(b)|$.This is contradiction. Therefore $(a, b) = (x, y)$ and h is injective. □

Theorem

Let $f : E \longrightarrow X$ be a function. If E is countable, then $f(E)$ is countable.

Proof.

Consider the equivalence relation $\sim$ in E defined by $x \sim y$ iff $f(x) = f(y)$. For each $x \in E$, let $C_x = \{y \in E : \ y \sim x\}$. The collection of these equivalence classes partitions E. Select an element a in each equivalence class C_x to obtain a set $A \subset E$ such that if $a, b \in A$ and $a \neq b$, then $f(a) \neq f(b)$. The set A is countable (subset of a countable set).The restriction $f_A : A \longrightarrow X$ of f is injective. Also $f_A : A \longrightarrow f_A(A) = f(E)$ is a bijection. Therefore $f(E)$ is equipotent to the countable set A □

Theorem

A countable union of countable sets is countable. More precisely, Let Λ be a countable set. For each $\lambda \in \Lambda$, let E_λ be a countable set. Then $E = \bigcup_{\lambda \in \Lambda} E_\lambda$ is countable.

Proof.

Suppose $E \neq \emptyset$ and Λ countably infinite (the case Λ finite is left as an exercise). We can write $\Lambda = \{\lambda_n : \ n \in \mathbb{N}\}$. By hypothesis for each $n \in \mathbb{N}$, E_{λ_n} is countable. If E_{λ_n} is finite with cardinality $|E_{\lambda_n}| = N(n)$, then there exists a bijection $f_n : \{1, \cdots, N(n)\} \longrightarrow E_{\lambda_n}$. Otherwise there exists a bijection $f_n : \mathbb{N} \longrightarrow E_{\lambda_n}$.

Consider the set

$$S = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \ E_{\lambda_n} \neq \emptyset \text{ and if } E_{\lambda_n} \text{ is finite} 1 \leq k \leq |E_{\lambda_n}|\}.$$

Now define the function

$$g : \ S \longrightarrow E = \bigcup_{\lambda_n \in \Lambda} E_{\lambda_n}$$

by $g(n,k) = f_n(k) \in E_{\lambda_n}$.The function g is surjective: If $x \in E$, then the exists $n \in \mathbb{N}$ such that $x \in E_{\lambda_n}$ and therefore $x = f_n(k) = g(n,k)$ for some $k \in \mathbb{N}$. By the previous theorem E is countable as the image via g of the countable set $S \subset \mathbb{N} \times \mathbb{N}$. □

Theorem

The set of rational numbers $\mathbb{Q}$ is countable.

Proof.

Let $\mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$ and $\mathbb{Q}^- = \mathbb{Q} \cap \mathbb{R}^-$ so that $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$. To prove that $\mathbb{Q}$ is countable it is enough to prove that $\mathbb{Q}^\pm$ is countable.The function $f : \mathbb{N} \times \mathbb{N} \longrightarrow Q^+$ given by $f(p, q) = p/q$ is surjective. Therefore $\mathbb{Q}^+$ is countable as the image of a countable set. A similar argument shows that $\mathbb{Q}^-$ is also countable. □

Theorem

Let $a, b \in \mathbb{R}$ with $a < b$. The interval $I = (a,\ b)$ is uncountable.

Proof.

By contradiction. Suppose that the interval $I = (a,\ b)$ is countable. Let $f : \mathbb{N} \longrightarrow I$ be a bijection. We have $a < f(1) < b$ and we can find an interval $[x_1,\ y_1] \subset (f(1),\ b)$ so that $f(1) \notin [x_1,\ y_1]$. Similarly for $f(2) \in I$, we can find an interval $[x_2,\ y_2] \subset [x_1,\ y_1]$ such that $f(2) \notin [x_2,\ y_2]$ (if $f(2) \notin [x_1,\ y_1]$ we can select $x_2,\ y_2$ arbitrary such that $x_1 < x_2 < y_2 < y_1$; if not, select $x_2,\ y_2$ arbitrary such that either $f(2) < x_2 < y_2 < y_1$ or $x_1 < x_2 < y_2 < f(2)$). This process can be continued by induction to produce a countable collection of nested intervals $\{[x_n,\ y_n]\}_{n\in\mathbb{N}}$ such that for every n, $f(n) \notin [x_n,\ y_n]$ and $[x_n,\ y_n] \subset [x_{n-1},\ y_{n-1}]$. The set $A = \{x_n,\ n \in \mathbb{N}\}$ is bounded above by y_k (for any $k \in \mathbb{N}$). Let $x^* = \sup(A)$. Then $x_n < x^* < y_n$ for every $n \in \mathbb{N}$ and so $x^* \in I$. Therefore, there exists $m \in \mathbb{N}$ such that $x^* = f(m)$. This implies $f(m) \in [x_m,\ y_m]$. A contradiction.

□