Real Analysis MAA 6616
Lecture 10
Approximation by Simple Functions

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions defined on a set E.

- The sequence $\{f_n\}$ converges to f pointwise on a set A if for every $x \in A$ we have $\lim_{n\to\infty} f_n(x) = f(x)$.
- The sequence $\{f_n\}$ converges to f pointwise a.e. on a set A if there exists a set Z of measure zero such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in A\backslash Z$.
- The sequence $\{f_n\}$ converges to f uniformly on a set A if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n - f| < \epsilon$ on A for all $n \geq N$.
- $\sup_n f_n$ is defined on E by $\sup_n f_n(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ and $\inf_n f_n$ is defined on E by $\inf_n f_n(x) = \inf\{f_n(x) : n \in \mathbb{N}\}$.
- $\overline{\lim}_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = \inf_{m\in\mathbb{N}}\{\sup_{n\geq m} f_n\}$
- $\underline{\lim}_{n\to\infty} f_n = \liminf_{n\to\infty} f_n = \sup_{m\in\mathbb{N}}\{\inf_{n\geq m} f_n\}$

Theorem (1)

Let $\{f_n\}$ be a sequence of measurable functions on a set $E \subset \mathbb{R}^q$. Suppose that f_n is finite a.e. on E for each $n \in \mathbb{N}$. Then

1. *Each function $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ is measurable on E.*
2. *If $\{f_n\}$ converges pointwise a.e. on E to a function f. Then f is measurable on E.*

Proof.

1. Let $c \in \mathbb{R}$. We have $\sup_n f_n(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$. Hence $\{\sup_n f_n > c\} = \bigcup_{n\in\mathbb{N}}\{f_n > c\}$ is measurable as countable union of measurable sets. Therefore $\sup_n f_n$ is measurable. Since $-f_n$ is measurable and $\inf_n f_n = -\sup_n(-f_n)$, then $\inf_n f_n$ is measurable.
   We have $\limsup_{n\to\infty} f_n = \inf_n F_n$ with $F_n = \sup\{f_p : p \geq n\}$. It follows that F_n is measurable for all n . Therefore $\inf_n F_n$ is also measurable. Similarly $\liminf_{n\to\infty} f_n = \sup_n G_n$, with $G_n = \inf\{f_p : p \geq n\}$, is measurable.

2. We know from part 1 that $\limsup_n f_n$ is measurable. If in addition $\{f_n\}$ converges to f a.e. on E, then $f = \limsup_n f_n$ a.e. on E and therefore f is measurable.

□

Let $E \subset \mathbb{R}^q$. A function $\phi : E \longrightarrow \mathbb{R}$ is said to be simple if it takes only finitely many distinct value: $\phi(E)$ is a finite subset of $\mathbb{R}$. In most statements we take simple functions to be also measurable.

Lemma (1)

If $\phi : E \longrightarrow \mathbb{R}$ is a simple and measurable function, then there exist a family of disjoint measurable sets $E_1, \cdots, E_n$ contained in E such that $E = E_1 \cup \cdots \cup E_n$, and there exist real numbers $a_1, \cdots, a_n$ such that

$$\phi = a_1\chi_{E_1} + a_2\chi_{E_2} + \cdots + a_n\chi_{E_n} = \sum_{j=1}^{j=n} a_j\chi_{E_j}$$

The Proof is left as an exercise.
The representation of ϕ given in Lemma 1 is called the canonical representation of a simple function

Lemma (2)

Let $f : E \longrightarrow \mathbb{R}$ be measurable and bounded. For every $\epsilon > 0$, there exist simple functions ϕ_ϵ and ψ_ϵ on E such that $0 \leq \psi_\epsilon - \phi_\epsilon \leq \epsilon$ and $\phi_\epsilon \leq f \leq \psi_\epsilon$ on the set E.

Proof.

Since f is bounded, then there exists an interval $(a,\ b)$ such that $f(E) \subset (a,\ b)$. For the given $\epsilon > 0$, we can find $n \in \mathbb{N}$ such that $\delta = \dfrac{b-a}{n} < \epsilon$. Consider the partition of $[a,\ b]$ given by $c_0 = a < c_1 < \cdots < c_n = b$ with $c_j = c_0 + j\delta$ for $j = 1, \cdots, n$. We have $c_j - c_{j-1} = \delta$. For each $j = 1, \cdots, n$, let $E_j = f^{-1}([c_{j-1},\ c_j))$. Then the collection $E_1, \cdots, E_n$ of measurable subset E is disjoint and covers E.
Define the simple functions ϕ_ϵ and ψ_ϵ on E by

$$\phi_\epsilon = \sum_{j=1}^{j=n} c_{j-1}\,\chi_{E_j} \quad \text{and} \quad \psi_\epsilon = \sum_{j=1}^{j=n} c_j\,\chi_{E_j}\ .$$

We have then

$$\psi_\epsilon - \phi_\epsilon = \sum_{j=1}^{j=n} (c_j - c_{j-1})\chi_{E_j} = \sum_{j=1}^{j=n} \delta\chi_{E_j} = \delta < \epsilon\ .$$

Also for any $x \in E$ there is a unique $j \in \{1, \cdots, n\}$ such that $x \in E_j$ and so $f(x) \in [c_{j-1},\ c_j)$ therefore $\phi_\epsilon(x) \leq f(x) < \psi_\epsilon(x)$. □

Before we state the main theorem, note that if ϕ and ψ are simple functions on $E \subset \mathbb{R}^q$, then $a\phi + b\psi$ is a simple function for any $a, b \in \mathbb{R}$. Indeed, if $\phi = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$ and $\psi = \sum_{k=1}^{m} \beta_k \chi_{B_k}$, then $a\phi + b\psi = \sum_{k=1}^{m} \sum_{j=1}^{n} (a\alpha_j + b\beta_k) \chi_{A_j \cap B_k}$.

Theorem (2)

Let $E \subset \mathbb{R}^q$ be a measurable set and $f : E \longrightarrow \overline{\mathbb{R}}$. Then

1. *If $f \geq 0$, then there is a sequence $\{\phi_n\}$ of simple functions defined on E such that $\{\phi_n\}$ converges pointwise to f. The sequence $\{\phi_n\}$ can be chosen to be increasing. That is $\phi_n \leq \phi_{n+1}$ for all $n \in \mathbb{N}$.*
2. *For an arbitrary function f that may change sign, there is a sequence $\{\phi_n\}$ of simple functions defined on E such that $\{\phi_n\}$ converges pointwise to f,*
3. *If f is measurable, then the ϕ_n's can be taken to be measurable.*
4. *If f is bounded, the convergence is uniform.*

Proof.

1. Let $n \in \mathbb{N}$. Divide the interval $[0,\ n]$ into $n2^n$ subintervals of equal length $1/2^n$ by the points $j/2^n$ with $j = 0, \cdots, n2^n$. Considers the collection of subsets of E defined by $A_{j,n} = f^{-1}\left([\frac{j-1}{2^n},\ \frac{j}{2^n})\right)$ for $j = 1, \cdots, n2^n$ and $B_n = f^{-1}([n,\ \infty])$. Note that this collection of subsets is disjoint and covers E. Define the simple function ϕ_n by

$$\phi_n = \sum_{j=1}^{n2^n} \frac{j-1}{2^n} \chi_{A_{j,n}} + n\chi_{B_n}\ .$$

The sequence of simple functions $\{\phi_n\}$ is increasing (exercise) and converges to f. Indeed, let $x \in E$. If $f(x) \neq \infty$, then there exits $m \in \mathbb{N}$ such that $f(x) < m$ and so for every $n \geq m$ there is a unique $j \in \{1, \cdots, n2^n\}$ such that $x \in A_{j,n}$. Hence $\frac{j-1}{2^n} \leq f(x) < \frac{j}{2^n}$ and $\phi_n(x) = \frac{j-1}{2^n}$. Therefore

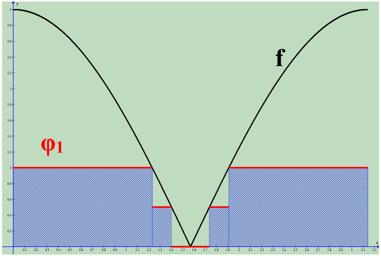
$$0 \leq f(x) - \phi_n(x) < \frac{1}{2^n} \quad \text{for all } n \geq m$$

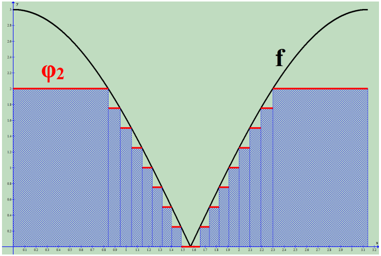
and so $\{\phi_n(x)\}$ converges to $f(x)$. If $f(x) = \infty$, then $x \in B_n$ and $\phi_n(x) = n$ converges to ∞.

2. Suppose that f changes sign. Consider the nonnegative functions $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. We have $f = f^+ - f^-$. By part 1 there are sequences of simple functions $\{\phi_n^+\}$ and $\{\phi_n^-\}$ that converge to f^+ and f^-. The sequence of simple functions $\{\phi_n^+ - \phi_n^-\}$ converges to f.

$\square$

Parts 3 and 4 are left as exercises


φ1
f


φ2
f