# Real Analysis MAA 6616
# Lecture 11
# Egorov and Luzin Theorems

## Egorov's Theorem

Egorov's Theorem states that if a sequence of measurable functions converges pointwise a.e. on a set of finite measure to a function that is a.e. finite, then it converges uniformly except on a subset with arbitrarily small measure.

Start with an example. Consider the sequence of piecewise linear functions $\{f_n\}$ defined on $[0,\ 2]$ as follows:

- For $n$ even:

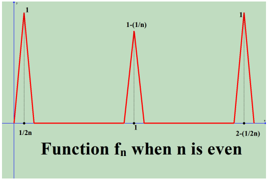
$$f_n(0) = f_n(\frac{1}{n}) = f_n(1 - \frac{1}{2n}) = f_n(1 + \frac{1}{2n}) = f_n(2 - \frac{1}{n}) = f_n(2) = 0;$$
$$f_n(\frac{1}{2n}) = f_n(2 - \frac{1}{2n}) = 1 \ \text{ and } \ f(1) = 1 - \frac{1}{n}$$

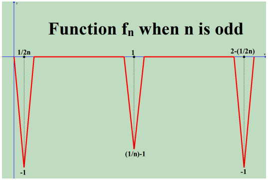
  and $f_n$ is the linear function connecting any two consecutive points so that $f(x) = 2nx$ for $x \in [0,\ 1/2n]$;

- For $n$ odd:

$$f_n(0) = f_n(\frac{1}{n}) = f_n(1 - \frac{1}{2n}) = f_n(1 + \frac{1}{2n}) = f_n(2 - \frac{1}{n}) = f_n(2) = 0;$$
$$f_n(\frac{1}{2n}) = f_n(2 - \frac{1}{2n}) = -1 \ \text{ and } \ f(1) = -1 + \frac{1}{n}$$

  and $f_n$ is the linear function connecting any two consecutive points so that $f(x) = -2nx$ for $x \in [0,\ 1/2n]$.




The sequence $\{f_n\}$ converges pointwise on $[0,\ 1) \cup (1,\ 2]$ to the function $f = 0$ and the sequence diverges for $x = 1$.
Indeed $f_{2j}(1) = 1 - (1/(2j))$ converges to 1 while $f_{2j+1}(1) = -1 + (1/(2j+1))$ converges to $-1$. If $x = 0$ or $x = 2$, then $f_n(x) = 0$ and the sequence converges to 0. If $x \neq 0, 1, 2$, let $\delta = \min(|x|, |1-x|, |2-x|)$. Let $N \in \mathbb{N}$ such that $N \geq \dfrac{1}{\delta}$. Then for every $n \geq N$ we have $f_n(x) = 0$ and $f_n(x)$ converges to 0.
Now for any $\epsilon > 0$ (no matter how small), let $N \geq \dfrac{1}{4\epsilon}$, then $\{f_n\}$ converges uniformly to $f = 0$ on the set $[\frac{\epsilon}{4},\ 1 - \frac{\epsilon}{4}] \cup [1 + \frac{\epsilon}{4},\ 2 - \frac{\epsilon}{4}]$ with measure $2 - \epsilon$.

## Lemma (1)

*Let $\{f_n\}$ be a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^q$ with finite measure. Assume that $\{f_n\}$ converges pointwise a.e. on $E$ to a function $f$ such that $f$ is finite a.e. on $E$. Then for every $\epsilon > 0$ and $\eta > 0$, there exists a measurable set $A \subset E$ and an integer $N \in \mathbb{N}$ such that $m(E \backslash A) < \eta$ and $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$ and for all $n > N$.*

## Proof.

Let $Z_1$ be the subset of $E$ where $f$ is not finite and let $Z_2$ be the subset of $E$ where $\{f_n\}$ fails to converge to $f$. Let $Z = Z_1 \cup Z_2$. Then $m(Z) = 0$ by hypothesis. For every $j \in \mathbb{N}$, let $A_j = \{x \in E \backslash Z : \ |f(x) - f_k(x)| < \epsilon \text{ for all } k \geq j\}$. The set $A_j$ is measurable since it can be expressed as a countable intersection of measurable sets: $A_j = \bigcap_{k \geq j} \{|f - f_k| < \epsilon\} \cap (E \backslash Z)$.

Note that $A_j \subset A_{j+1}$ and $\bigcup_{j \geq 1} A_j = E \backslash Z$ (since $f_n \longrightarrow f$ on $E \backslash Z$). It follows from the continuity of the Lebesgue measure that $\lim_{n \to \infty} m(A_n) = m(E \backslash Z) = m(E)$. Therefore for the given $\eta > 0$ there exists $N \in \mathbb{N}$ such that $m(E \backslash A_N) < \eta$ and for every $x \in A = A_N$ and for every $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$. □

## Theorem (1)

*Let $\{f_n\}$ be a sequence of measurable functions on a measurable set $E \subset \mathbb{R}^q$ with finite measure. Assume that $\{f_n\}$ converge pointwise a.e. on $E$ to a function $f$ such that $f$ is finite a.e. on E. Then for every $\eta > 0$ there exists a closed set $A \subset E$ such that $m(E \backslash A) < \eta$ and $\{f_n\}$ converges uniformly to $f$ on A.*

## Proof.

Let $m \in \mathbb{N}$. It follows from Lemma 1 that there exists a measurable set $A_m \subset E$ and $N(m) \in \mathbb{N}$ such that $m(E \backslash A_m) < \dfrac{\eta}{2^{m+1}}$ and $|f_n - f| < \dfrac{1}{m}$ for all $n \geq N(m)$.

Let $\tilde{A} = \bigcap_{m=1}^{\infty} A_m$. The set $\tilde{A}$ is measurable and

$$m(E \backslash \tilde{A}) = m\left(\bigcup_{m=1}^{\infty} (E \backslash A_m)\right) \leq \sum_{m=1}^{\infty} \frac{\eta}{2^{m+1}} = \frac{\eta}{2}\,.$$

Now we prove that $\{f_n\}$ converges uniformly to $f$ on $\tilde{A}$. Let $\epsilon > 0$ and let $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. It follows from the definition of $\tilde{A}$ and of $A_m$ that for every $n \geq N(m)$ and for every $x \in \tilde{A} \subset A_m$ we have $|f_n - f| < \dfrac{1}{m} < \epsilon$. This proves the uniform convergence on $\tilde{A}$

Next, since $\tilde{A}$ is measurable we can find a closed set $A \subset \tilde{A}$ such that $m(\tilde{A} \backslash A) < \eta/2$. The sequence is uniformly convergent on $A$ and

$$m(E \backslash A) = m\left((E \backslash \tilde{A}) \cup (\tilde{A} \backslash A)\right) \leq m(E \backslash \tilde{A}) + m(\tilde{A} \backslash A) \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

□

Luzin's Theorem states that a measurable function on $E$ is "nearly continuous" in the following sense: For any $\epsilon > 0$, there is a subset whose measure is within $\epsilon$ to that of $E$ and on which the function is continuous. We start with the case of simple functions.

## Proposition (1)

*Let $\phi$ be a simple and measurable function defined on a set E. Then for every $\epsilon > 0$ there exists a closed $F \subset E$ such that $m(E\backslash F) < \epsilon$ and $\phi$ continuous on F.*

## Proof.

We use the canonical representation of simple function to express $\phi$ as $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ for some collection of $n$ disjoint measurable sets $E_1, \cdots, E_n$ in $E$. For $\epsilon > 0$, there exist closed sets $F_1, \cdots, F_n$ with $F_j \subset E_j$ and $m(E_j \backslash F_j) < \epsilon/n$. Then $F = \bigcup_{j=1}^{n} F_j$ is a closed subset in $E$ and $m(E\backslash F) < \epsilon$. Note that the restriction of $\phi$ to each $F_j$ is continuous since it is constant on each $F_j$. It remains to verify that $\phi$ is continuous on their disjoint union $F$.

For $N \in \mathbb{N}$, let $B_N = B_N(0)$ be the ball in $\mathbb{R}^q$ with center 0 and radius $N$. Let $F_j^N = F_j \cap B_N$ and $F^N = \bigcup_{j=1}^{n} F_j^N$. To prove the continuity of $\phi$ on $F$ it is enough to prove continuity on $F^N$ for an arbitrary $N$ (since $F = \bigcup_N F^N$). To prove continuity on $F^N$, it is enough to verify that the closed sets $F_j^N$'s are separated. That is, there exists $\delta_0 > 0$ such that for $j \neq k$, and for every $x \in F_j^N$ and $y \in F_k^N$ we have $|x - y| \geq \delta_0$.

For $j \neq k$, let $\delta_{j,k} = \inf\{|x - y| : x \in F_j^N, y \in F_k^N\}$. We claim that $\delta_{j,k} > 0$. Indeed, if $\delta_{j,k} = 0$, then there would be sequences $\{x_p\}_p \subset F_j^N$ and $\{y_p\}_p \subset F_k^N$ such that $\lim_{p \to \infty} |x_p - y_p| = 0$. Since $F_j^N$ and $F_k^N$ are compact, then we can extract convergent subsequences $\{x_{p_m}\}_m \subset F_j^N$ and $\{y_{p_m}\}_m \subset F_k^N$ that converge to the same limit $z \in F_j^N \cap F_k^N$ (since $\delta_{j,k} = 0$) and this is a contradiction since $F_j$ and $F_k$ are disjoint. Hence $\delta_{j,k} > 0$ and $\delta_0 = \min\{\delta_{j,k} : j \neq k\}$ is strictly positive. Therefore the $F_j^N$'s are separated and $\phi$ is continuous on $F^N$. □

## Theorem (2)

*Let $f : E \subset \mathbb{R}^q \longrightarrow \overline{\mathbb{R}}$ be measurable and finite a.e. on E. Then for every $\epsilon > 0$, there exists a closed $F \subset E$ such that $m(E \backslash F) < \epsilon$ and $f$ continuous on $F$.*

## Proof.

Since $f$ is measurable, by the Simple Approximation Theorem, there exists a sequence $\{\phi_n\}$ of simple functions on $E$ which converges pointwise on $E$ to $f$. According to Egorov's Theorem, given any $\epsilon > 0$, there exists a closed set $C \subset E$ with $m(E \backslash C) < \epsilon/2$ and $\{\phi_n\}$ converges uniformly on $C$. For each $n \in \mathbb{N}$, there exists a closed set $F_n \subset E$ with $m(E \backslash F_n) < (\epsilon/2^{n+1})$ such that the simple function $\phi_n$ is continuous on $F_n$. Consider the closed set $F = C \cap \left(\bigcap_{n=1}^{\infty} F_n\right)$. We have $E \backslash F = (E \backslash C) \cup \left(\bigcup_{n=1}^{\infty} (E \backslash F_n)\right)$ so that

$$m(E \backslash F) \leq m(E \backslash C) + \sum_{n=1}^{\infty} m\left(E \backslash F_n\right) \leq \frac{\epsilon}{2} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon \,.$$

Since each function $\phi_n$ is continuous on $F \subset F_n$ and $\{\phi_n\}$ converges uniformly on $F$, then the limit $f$ is also continuous on $F$. □