Real Analysis MAA 6616
Lecture 12
The Riemann Integral and
The Lebesgue Integral for Simple Functions

The Riemann Integral

Let $f : B \subset \mathbb{R}^q \longrightarrow \mathbb{R}$ be a bounded function defined on the bounded box in $B = I_1 \times \cdots \times I_q$, where I_j $(j = 1, \cdots, q)$ is a closed interval in $\mathbb{R}$: $I_j = [a_j,\ b_j]$. The Riemann Integral of f over B denoted

$$(R) \int_{a_1}^{b_1} \cdots \int_{a_q}^{b_q} f(x) dx_q \cdots dx_1 \text{ or simply } (R) \int_B f(x) dx$$

is defined as follows: Partition B into a finite collection $\mathcal{P} = \{B_k\}_{k=1}^N$ of nonoverlapping boxes so that $B = \bigcup_{k=1}^N B_k$ (two boxes are nonoverlapping if $B_j \cap B_k$ has an empty interior for $j \neq k$) and $\text{vol}(B) = \sum_{k=1}^N \text{vol}(B_k)$, where $\text{vol}(B) = m(B)$ is the measure or volume of B. Let

$$m_k = \inf\{f(x) :\ x \in B_k\} \text{ and } M_k = \sup\{f(x) :\ x \in B_k\}\,.$$

The Lower and Upper Darboux Sums associated with the partition $\mathcal{P}$ are

$$L(f, \mathcal{P}) = \sum_{k=1}^N m_k \text{vol}(B_k) \text{ and } U(f, \mathcal{P}) = \sum_{k=1}^N M_k \text{vol}(B_k)\,.$$

The Lower and Upper Riemann Integrals of f over B are:

$$(R) \underline{\int}_B f(x) dx = \sup\{L(f, \mathcal{P}) :\ \mathcal{P} \text{ partition of } B\}$$

$$(R) \overline{\int}_B f(x) dx = \inf\{U(f, \mathcal{P}) :\ \mathcal{P} \text{ partition of } B\}$$

Since f is bounded, then the lower and upper Riemann integrals are finite and

$$(R)\underline{\int_B} f(x)dx \leq (R)\overline{\int_B} f(x)dx\,.$$

The function f is said to be Riemann integrable if the lower and upper Riemann integrals are equal and call this common value the Riemann integral of f and denote it $\int_B f(x)dx$

Note that if ϕ is a simple function with canonical representation $\phi = \sum_{k=1}^N \alpha_k \chi_{B_k}$, then

$$L(\phi, \mathcal{P}) = \sum_{k=1}^{N} \alpha_k \,\mathrm{vol}(B_k) = U(\phi, \mathcal{P}) = (R)\int_B \phi(x)\,dx\,.$$

It follows that the lower and upper Riemann integral of a function f can be rewritten as:

$$(R)\underline{\int_B} f(x)dx = \sup\{(R)\int_B \phi(x)\,dx : \ \phi \text{ simple function and } \phi \leq f\}$$

$$(R)\overline{\int_B} f(x)dx = \inf\{(R)\int_B \phi(x)\,dx : \ \phi \text{ simple function and } \phi \geq f\}$$

Theorem (1)

If $f : B \subset \mathbb{R}^q \longrightarrow \mathbb{R}$ is continuous on the box B, then f is Riemann integrable over B.

Proof.

Since B is compact, then f is uniformly continuous on B. Let $\epsilon > 0$. There exists $\delta > 0$ such that for every $x, y \in B$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \dfrac{\epsilon}{\text{vol}(B)}$. Consider a partition $\mathcal{P} = \{B_k\}_{k=1}^{N}$ of B such that $|x - y| < \delta$ for every $x, y \in B_k$ for all $k = 1, \cdots, N$. It follows that for every $k = 1, \cdots, N$ we have

$$M_k - m_k = \sup\{f(x) : x \in B_k\} - \inf\{f(x) : x \in B_k\} \leq \frac{\epsilon}{\text{vol}(B)} .$$

Hence

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{k=1}^{N} (M_k - m_k)\text{vol}(B_k) \leq \sum_{k=1}^{N} \frac{\epsilon}{\text{vol}(B)} \text{vol}(B_k) \leq \epsilon .$$

□

Remark (1)

There are non continuous functions that are Riemann integrable. For example, let $\{B_k\}_{k=1}^{N}$ be a covering of B by nonoverlapping boxes and let $\alpha_1, \cdots, \alpha_N$ be real number, then $f = \sum_{k=1}^{N} \alpha_k \chi_{B_k}$ is not necessarily continuous but it is Riemann integrable.

Remark (2)

The Dirichlet function on $[0, 1]$ is defined by $f = \chi_{\mathbb{Q} \cap [0, 1]}$. That is $f(x) = 1$ if $x \in [0, 1]$ is a rational number and $f(x) = 0$ if $x \in [0, 1]$ is an irrational number. The Dirichlet function is not Riemann integrable. Indeed it follows from the density of Q in $\mathbb{R}$ that for every partition $\mathcal{P}$ of $[0, 1]$, $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) = 1$.

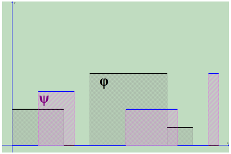
Let ϕ be a measurable nonnegative simple function defined on a measurable set $E \subset \mathbb{R}^q$ with finite measure. Thus there exists a partition of E by a finite collection of disjoint measurable sets $\{E_j\}_{j=1}^n$ and real numbers $a_j \geq 0$ such that $\phi = \sum_{j=1}^n a_j \chi_{E_j}$. The Lebesgue integral of ϕ over E is defined as:

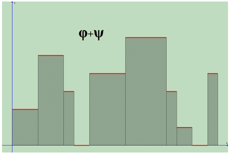
$$\int_E \phi \, dx = \sum_{j=1}^n a_j \, m(E_j) \, .$$

Proposition (1)

Let ϕ and ψ be nonnegative measurable simple functions on a measurable set E. Then

$$\int_E (\phi + \psi) dx = \int_E \phi dx + \int_E \psi dx \, .$$





Proof.

There are partitions of E by finite collections of disjoints measurable sets $\{A_j\}_{j=1}^N$ and $\{B_k\}_{k=1}^M$ and nonnegative real numbers $\{a_j\}_{j=1}^N$ and $\{b_k\}_{k=1}^M$ such that $\phi = \sum_{j=1}^{N} a_j \chi_{A_j}$ and $\psi = \sum_{k=1}^{M} b_k \chi_{B_k}$. Since $A_j = \bigcup_{k=1}^M (A_j \cap B_k)$ and $B_k = \bigcup_{j=1}^N (A_j \cap B_k)$, then the function $\phi + \psi$ can be written as

$$\phi + \psi = \sum_{j=1}^{N} \sum_{k=1}^{M} (a_j + b_k) \chi_{A_j \cap B_k} .$$

Therefore

$$\begin{aligned}
\int_E (\phi + \psi) dx &= \sum_{j=1}^{N} \sum_{k=1}^{M} (a_j + b_k)\, m(A_j \cap B_k) \\
&= \sum_{j=1}^{N} a_j \sum_{k=1}^{M} m(A_j \cap B_k) \;+\; \sum_{k=1}^{M} b_k \sum_{j=1}^{N} m(A_j \cap B_k) \\
&= \sum_{j=1}^{N} a_j\, m(A_j) \;+\; \sum_{k=1}^{M} b_k\, m(B_k) \\
&= \int_E \phi\, dx \;+\; \int_E \psi\, dx
\end{aligned}$$

□

Proposition (2)

Let ϕ be a nonnegative measurable simple functions on a measurable set E and $\lambda > 0$. Then $\displaystyle\int_E \lambda \phi dx = \lambda \int_E \phi dx$

Note that if ϕ is a nonnegative measurable simple function on the measurable set E and if $F \subset E$ is a measurable subset, then the restriction of $\phi|_F$ to F is a nonnegative simple function and define $\int_F \phi dx = \int_F \phi_{|_F} dx$.

Proposition (3)

Let ϕ and be a nonnegative measurable simple functions on a measurable set E. Then

- $\int_E \phi dx = 0$ *if and only if* $\phi = 0$ *a.e.*
- *Let $F \subset E$ be a measurable subset, then $\phi\chi_F$ is a nonnegative simple function and* $\int_E \phi\chi_F \, dx = \int_F \phi \, dx$.
- *Let E_1, E_2 be disjoint measurable subsets of E such that $E = E_1 \cup E_2$. Then*

$$\int_E \phi \, dx = \int_{E_1} \phi \, dx + \int_{E_2} \phi \, dx$$

Proposition (4)

Let ϕ and ψ be nonnegative simple functions on a measurable set E. If $\phi \leq \psi$, then $\int_E \phi dx \leq \int_E \psi dx$

Proof.

In this case $\psi - \phi$ is a nonnegative simple function and $\psi = (\psi - \phi) + \phi$. Proposition 1 implies $\int_E \phi dx \leq \int_E \psi dx$ □

Proposition (5)

Let ϕ and be a nonnegative simple functions on a measurable set E.

1. *Let $\{A_k\}_{k=1}^{\infty}$ be a collection of disjoint measurable sets such that $E = \bigcup_{k=1}^{\infty} A_k$. Then*

$$\int_E \phi \, dx = \sum_{k=1}^{\infty} \int_{A_k} \phi \, dx$$

2. *Let $\{A_k\}_{k=1}^{\infty}$ be an ascending collection of measurable sets ($A_n \subset A_{n+1}$) such that $E = \bigcup_{k=1}^{\infty} A_k$. Then*

$$\int_E \phi \, dx = \lim_{n \to \infty} \int_{A_n} \phi \, dx$$

Proof.

Let $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$ be the canonical representation of ϕ. The set E is the disjoint union of $E_1, \cdots, E_N$. For each $j = 1, \cdots, N$ let $A_{k,j} = A_k \cap E_j$. The collection $\{A_{k,j}\}_{k=1}^{\infty}$ consists of disjoint measurable sets such that $E_j = \bigcup_{k=1}^{\infty} A_{k,j}$. We have $m(E_j) = \sum_{k=1}^{\infty} m(A_{k,j})$. Therefore

$$\int_E \phi dx = \sum_{j=1}^{N} a_j \, m(E_j) = \sum_{j=1}^{N} a_j \left(\sum_{k=1}^{\infty} m(A_{k,j}) \right) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{N} a_j m(A_{k,j}) \right) = \sum_{k=1}^{\infty} \int_{A_j} \phi \, dx$$

Part 2 is left as an exercise. □