Real Analysis MAA 6616
Lecture 13
The Lebesgue Integral of a Bounded Function Over a Set of Finite Measure

Let $\phi = \sum_{j=1}^{N} a_j \chi_{E_j}$ be any measurable simple function on a measurable set E (so ϕ could take positive and negative values). Let $\phi^+ = \max(\phi, 0)$ and $\phi^- = \max(-\phi, 0)$. So that both ϕ^+ and ϕ^- are nonnegative simple functions on E and $\phi = \phi^+ - \phi^-$. Define the Lebesgue integral of ϕ as

$$\int_E \phi\, dx = \int_E \phi^+\, dx - \int_E \phi^-\, dx = \sum_{j=1}^{N} a_j\, m(E_j)\,.$$

Let E be a measurable set with finite measure and let $f : E \longrightarrow \mathbb{R}$ be a bounded function. Define the lower and upper Lebesgue integrals of f over E as:

$$\underline{\int_E} f(x)dx = \sup\left\{\int_E \phi(x)\, dx :\ \phi \text{ simple function and } \phi \leq f\right\}$$

$$\overline{\int_E} f(x)dx = \inf\left\{\int_E \psi(x)\, dx :\ \psi \text{ simple function and } \psi \geq f\right\}$$

Note that since f is bounded, then whenever the simple functions ϕ and ψ satisfy $\phi \leq f \leq \psi$ we have $\int_E \phi\, dx \leq \int_E \psi\, dx$ and it follows that $\underline{\int_E} f dx \leq \overline{\int_E} f dx$.
A bounded function f on a measurable set E with $m(E) < \infty$ is said to be Lebesgue integrable if $\underline{\int_E} f dx = \overline{\int_E} f dx$. The common value is the Lebesgue integral of f on E and is denoted $\int_E f dx$.

The following theorem follows directly from the definitions of the Riemann and Lebesgue integrals.

Theorem (1)

Let $E \subset \mathbb{R}^q$ be a bounded and measurable set and $f : E \longrightarrow \mathbb{R}$ be a bounded function. If f is Riemann integrable over E, then it is Lebesgue integrable over E.

Remark (1)

There exist Lebesgue integrable functions that are not Riemann integrable. For example, the Dirichlet function on $[0,\ 1]$ given by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational is not Riemann integrable (Lecture 12). However, since $f = \chi_E$ where $E = \mathbb{Q} \cap [0,\ 1]$ is measurable, we have $\int_{[0,\ 1]} f dx = m(E) = 0$.

Theorem (2)

Let $E \subset \mathbb{R}^q$ be a measurable set with finite measure and $f : E \longrightarrow \mathbb{R}$ be a bounded and measurable function. Then f is Lebesgue integrable.

Proof.

Let $\epsilon > 0$. It follows from the Simple Approximation Lemma (Lecture 10) that there exist simple functions ϕ_ϵ and ψ_ϵ on E such that $\phi_\epsilon \leq f \leq \psi_\epsilon$ and $\psi_\epsilon - \phi_\epsilon \leq \dfrac{\epsilon}{m(E)}$. Therefore

$$0 \leq \overline{\int}_E fdx - \underline{\int}_E fdx \leq \int_E \psi_\epsilon dx - \int_E \phi_\epsilon dx = \int_E (\psi_\epsilon - \phi_\epsilon) dx \leq \epsilon\,.$$

Since $\epsilon > 0$ is arbitrary we have $0 = \overline{\int}_E fdx - \underline{\int}_E fdx$ and f is Lebesgue integrable.

Theorem (3)

Let $f,\ g\ : E\ \longrightarrow\ \mathbb{R}$ be bounded and measurable on E with $m(E) < \infty$. Then

1. *(Linearity)* $\int_E (af + bg)\, dx = a\int_E f\, dx + b\int_E g\, dx$ *for any* $a,\ b\ \in \mathbb{R}$.
2. *(Monotonicity) If* $f \leq g$ *a.e. on* E*, then* $\int_E f\, dx\ \leq\ \int_E g\, dx$.

Proof.

1. The function $af + bg$ is measurable and bounded and so integrable. Observe that if $S \subset \mathbb{R}$ is bounded and $a > 0$, then $\sup(aS) = a\sup(S)$ and if $a < 0$, $\sup(aS) = a\inf(S)$ where $aS = \{x = as \in \mathbb{R}\ :\ s \in S\}$. We first show that $\int_E af\, dx = a\int_E f\, dx$. Suppose $a > 0$. If ϕ and ψ are simple functions such that $\phi \leq af \leq \psi$, then $\frac{\phi}{a}$ and $\frac{\psi}{a}$ are simple functions and $\frac{\phi}{a} \leq f \leq \frac{\psi}{a}$.
   $a\int_E f\, dx = a\inf\left\{\int_E \psi dx\ :\ \psi \geq f\right\} = \inf\left\{\int_E \tilde{\psi} dx\ :\ \tilde{\psi} \geq af\right\} = \int_E af\, dx$
   For $a < 0$, we have $a\int_E f\, dx = a\inf\left\{\int_E \psi dx\ :\ \psi \geq f\right\} = \sup\left\{\int_E \tilde{\phi} dx\ :\ \tilde{\phi} \leq af\right\} = \int_E af\, dx$
   We are left to prove $\int_E (f + g)dx = \int_E fdx + \int_E gdx$. Let $\epsilon > 0$. There are simple functions ϕ_1 and ϕ_2 such that $\phi_1 \leq f,\ \phi_2 \leq g,\ \int_E \phi_1 \geq \int_E fdx + \frac{\epsilon}{2}$ and $\int_E \phi_2 \geq \int_E 2dx + \frac{\epsilon}{2}$. Therefore

$$\int_E (f + g)dx \geq \int_E (\phi_1 + \phi_2)dx = \int_E \phi_1 dx + \int_E \phi_1 dx \geq \int_E fdx + \int_E gdx + \epsilon.$$

   Similarly we can show by using upper simple functions that $\int_E (f + g)dx \leq \int_E fdx + \int_E gdx + \epsilon$. Since ϵ is arbitrary we have $\int_E (f + g)dx = \int_E fdx + \int_E gdx$

2. The function $h = g - f$ is measurable and $h \geq 0$ a.e. on E. Hence for any simple function $\psi \geq h$ we have $\psi \geq 0$ a.e. and so $\int_E \psi dx \geq 0$. It follows that $\int_E hdx \geq$. The linearity of the integral implies that $\int_E gdx \geq \int_E fdx$.

Theorem (4)

Let $E \subset \mathbb{R}^q$ be a measurable set with finite measure and $f : E \longrightarrow \mathbb{R}$ be a bounded and measurable function. Let $A \subset E$ and $B \subset E$ be disjoint and measurable. Then

$$\int_{A\cup B} fdx = \int_A fdx + \int_B fdx \,.$$

Proof.

First note that if $F \subset E$ is measurable and ϕ is a simple function on E, then $\phi\chi_F$ is a simple function on F (and on E) and $\int_F \phi dx = \int_E \phi\chi_F dx$. It follows that from this observation and the definition of the Lebesgue integral that for any bounded measurable function f we have $\int_F fdx = \int_E f\chi_F dx$.
Since A and B are disjoint then $\chi_{A\cup B} = \chi_A + \chi_B$. We have then

$$\int_{A\cup B} fdx = \int_E f\chi_{A\cup B} dx = \int_E f(\chi_A + \chi_B) dx = \int_E f\chi_A dx + \int_E f\chi_B dx = \int_A fdx + \int_B fdx.$$

$\square$

Theorem (5)

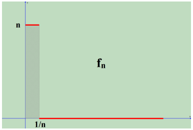
Let $E \subset \mathbb{R}^q$ be a measurable set with finite measure and $f : E \longrightarrow \mathbb{R}$ be a bounded and measurable function. Then

$$\left| \int_E f dx \right| \leq \int_E |f| \, dx \, .$$

Proof.

The function $|f|$ is bounded and measurable and $-|f| \leq f \leq |f|$. Therefore $-\int_E |f| \, dx \leq \int_E f dx \leq \int_E |f| \, dx.$ □

The following example shows that if $\{f_n\}_n$ is a sequence of measurable functions with $f_n \longrightarrow f$, then the sequence of integrals $\int_E f_n dx$ might not converge to $\int_E f dx$. Consider the sequence of functions $f_n(x)$ on $(0, 1)$ given by $f_n(x) = n$ for $0 < x < 1/n$ and $f_n(x) = 0$ for $1/n \leq x < 1$.



The sequence f_n converges pointwise to $f = 0$ but $\int_0^1 f_n dx = 1$ does not converge to $\int_0^1 0 \, dx = 0.$

Theorem (6)

Let $\{f_n\}$ be a sequence of bounded measurable functions on a set E with finite measure. Suppose that $\{f_n\}$ converges uniformly on E to a function f. Then $\lim_{n\to\infty} \int_E f_n dx = \int_E f dx.$

Proof.

Since the convergence is uniform and each f_n is bounded, then the limit f is bounded. Also f is measurable (since a pointwise limit of measurable functions is measurable). Hence f is integrable. Let $\epsilon > 0$. It follows from the uniform convergence that there exists $N > 0$ such that for every $n > N$ we have $|f_n - f| \leq \dfrac{\epsilon}{m(E)}$ on E. Therefore,

$$\left|\int_E f_n dx - \int_E f dx\right| = \left|\int_E (f_n - f) dx\right| \leq \int_E |f_n - f| \, dx \leq \epsilon. \qquad \square$$

Theorem (7: Bounded Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions on a set E with finite measure. Suppose that $\{f_n\}$ is uniformly bounded (i.e. there exists $M > 0$ such that $|f_n| \leq M$ for all $n \in \mathbb{N}$) and suppose that the sequence converges pointwise to f on E. Then $\lim_{n\to\infty} \int_E f_n dx = \int_E f dx.$

Proof.

First note that since $f_n \longrightarrow f$ pointwise and $|f_n| \leq M$ for all M, then $|f| \leq M$. Also since a pointwise limit of measurable functions is measurable, then f is measurable and so integrable.

Let $\epsilon > 0$. It follows from Egorov's Theorem that there exists a measurable set $F \subset E$ with $m(E \backslash F) < \dfrac{\epsilon}{4M}$ such that $f_n \longrightarrow f$ uniformly on F. Let $N \in \mathbb{N}$ such that $|f_n - f| < \dfrac{\epsilon}{2m(E)}$ for all $n > N$. We have

$$
\begin{aligned}
\left| \int_E f_n dx - \int_E f dx \right| &= \left| \int_E (f_n - f) dx \right| \\
&= \left| \int_F (f_n - f) dx + \int_{E \backslash F} f_n dx - \int_{E \backslash F} f dx \right| \\
&\leq \int_F |f_n - f| \, dx + \int_{E \backslash F} |f_n| \, dx + \int_{E \backslash F} |f| \, dx \\
&\leq \frac{\epsilon}{2m(E)} m(F) + M \, m(E \backslash F) + M \, m(E \backslash F) \leq \epsilon
\end{aligned}
$$

□