Real Analysis MAA 6616
Lecture 14
The Lebesgue Integral of Nonnegative Functions

In Lecture 13 we defined the Lebesgue integral of a **bounded** measurable function over a set of **finite** measure. Now we will consider integrals of measurable functions, not necessarily bounded, over measurable sets, not necessarily with finite measure. We start with nonnegative functions.

A measurable function $h : E \longrightarrow \overline{R}$ is said to have a finite support if there exists a set $E_0 \subset E$ with $m(E_0) < \infty$ such that $h = 0$ on $E \backslash E_0$. Define $\mathcal{M}_0(E)$ as the space of bounded and measurable functions with finite suport in E:

$$\mathcal{M}_0(E) = \{h : E \longrightarrow \mathbb{R} : \ h \text{ bounded}, \ \text{measurable, and with finite support}\}$$

Note that since h is bounded and with finite support $E_0 \subset E$, then

$$\int_{E_0} hdx = \int_E h\chi_{E_0}\,dx = \int_E hdx.$$

Let $E \subset \mathbb{R}^q$ be a measurable set and let $f : E \longrightarrow [0, \ \infty]$. Define the (Lebesgue) integral of f over E as:

$$\int_E fdx = \sup\left\{\int_E hdx : \ h \in \mathcal{M}_0(E) \text{ and } 0 \leq h \leq f\right\}$$

Note $\displaystyle\int_E fdx$ and could be ∞.

Theorem (1. Chebychev's Inequality)

Let $E \subset \mathbb{R}^q$ be a measurable set and $f : E \longrightarrow [0, \infty]$ be a measurable function. Then for every $\lambda > 0$

$$m(\{f > \lambda\}) \leq \frac{1}{\lambda} \int_E f \, dx \, .$$

Proof.

Let $E_\lambda = \{f > \lambda\}$. consider two cases depending on wether the measure of E_λ is finite or not.

- Case $m(E_\lambda) < \infty$: The function $h = \lambda \chi_{E_\lambda} \in \mathcal{M}_0(E)$ (it is measurable, bounded, and with finite support) and $0 \leq h \leq f$. Therefore

$$\int_E h \, dx = \int_{E_\lambda} \lambda \, dx = \lambda m(E_\lambda) \leq \int_E f \, dx$$

  and the conclusion of the theorem follows.

- Case $m(E_\lambda) = \infty$: For $n \in \mathbb{N}$, let $B_n(0)$ be the ball centered at 0 with radius n and let $E_{\lambda,n} = E_\lambda \cap B_n(0)$. Then $E_{\lambda,n}$ is bounded and $\lim_{n\to\infty} m(E_{\lambda,n}) = m(E) = \infty$. The function $h_n = \lambda \chi_{E_{\lambda,n}} \in \mathcal{M}_0(E)$, $0 \leq h \leq f$, and $\int_E h_n \, dx = \lambda m(E_{\lambda,n})$. Therefore

$$\lambda m(E_\lambda) = \infty = \lim_{n\to\infty} (\lambda m(E_{\lambda,n})) = \lim_{n\to\infty} \int_E h_n \, dx \leq \int_E f \, dx \, .$$

□

As a consequence of Chebychev's inequality we have

Proposition (1)

Let $E \subset \mathbb{R}^q$ be a measurable set and $f : E \longrightarrow [0, \infty]$ be a measurable function. Then

$$\int_E f\,dx = 0 \quad \Longleftrightarrow \quad f = 0 \ \text{ a.e. on } E$$

Proof.

"$\Longrightarrow$" Suppose that $\int_E f\,dx = 0$. It follows from Chebychev's inequality that for every $n \in \mathbb{N}$ we have

$$m\left(\{f > \frac{1}{n}\}\right) \leq n \int_E f\,dx = 0. \text{ Therefore } m\left(\{f > 0\}\right) = m\left(\bigcup_{n=1}^{\infty} \{f > \frac{1}{n}\}\right) \leq \sum_{n=1}^{\infty} m\left(\{f > \frac{1}{n}\}\right) = 0\,,$$

and $f = 0$ a.e.

"$\Longleftarrow$" Suppose that $f = 0$ a.e. on E. Let $h \in \mathcal{M}_0(E)$ such that $0 \leq h \leq f$. Then $h = 0$ a.e. on E. It follows that every simple function ϕ such that $0 \leq \phi \leq h$ satisfy $\phi = 0$ a.e. on E. Consequently $\int_E \phi\,dx = 0$. It follows from the definition of the Lebesgue integral that $\int_E h\,dx = 0$ and then $\int_E f\,dx = 0$. □

Theorem (2)

Let $E \subset \mathbb{R}^q$ be a measurable set and $f, g : E \longrightarrow [0, \infty]$ be measurable functions. Then

1. *Linearity:* $\int_E (af + bg)\,dx = a\int_E f\,dx + b\int_E g\,dx$ *for* $a \geq 0$ *and* $b \geq 0$.
2. *Monotonicity: If* $f \leq g$, *then* $\int_E f\,dx \leq \int_E g\,dx$

Proof.

1. Let $a > 0$. For $h \in \mathcal{M}_0(E)$ such that $0 \leq h \leq af$, it follows from the definition of the integral of nonnegative functions, as the supremum of the integrals of such functions h, that $\int_E \frac{h}{a} dx \leq \int_E f dx$ and so $\int_E h dx \leq a \int_E f dx$. This implies that $\int_E (af) dx \leq a \int_E f dx$. The inequality $a \int_E f dx \leq \int_E (af) dx$ is left as an exercise.
   Now we prove $\int_E (f+g) dx = \int_E f dx + \int_E g dx$. Let h, k be arbitrary functions in $\mathcal{M}_0(E)$ such that $0 \leq h \leq f$ and $0 \leq k \leq g$. Then $h + k \in \mathcal{M}_0(E)$ and $0 \leq h + k \leq f + g$. It follows from the definition of the Lebesgue integral of $f + g$ as a supremum that $\int_E (h+k) dx \leq \int_E (f+g) dx$. The linearity of the integral of functions with finite support gives $\int_E h dx + \int_E k dx \leq \int_E (f+g) dx$. The arbitrariness of $h, k \in \mathcal{M}_0(E)$ with $0 \leq h \leq f$ and $0 \leq k \leq g$ implies $\int_E f dx + \int_E g dx \leq \int_E (f+g) dx$. It remains to prove $\int_E f dx + \int_E g dx \geq \int_E (f+g) dx$.
   Let $\phi \in \mathcal{M}_0(E)$ such that $0 \leq \phi \leq (f+g)$. Define functions h and k by: $h = \min\{f, \phi\}$ and $k = \phi - h$. Then both $h, k \in \mathcal{M}_0(E)$. Indeed if the supports of h and k are contained in the support of ϕ. If $x \in E$ is not in the support of ϕ, then $\phi(x) = 0$ and $h(x) = \min\{\phi(x), f(x)\} = 0$ and $k(x) = \phi(x) - h(x) = 0$. Also both h and k are nonnegative and $h \leq f$ and $k = \phi - h \leq f + g - h \leq g$. We have then (linearity of integral for functions in $\mathcal{M}_0(E)$) $\int_E \phi dx = \int_E h dx + \int_E k dx \leq \int_E f dx + \int_E g dx$. Since ϕ is arbitrary in $\mathcal{M}_0(E)$ with $\phi \leq f + g$, then $\int_E (f+g) dx = \int_E f dx + \int_E g dx$.
2. Suppose that $f \leq g$. To prove $\int_E f\, dx \leq \int_E g\, dx$, it is enough to prove the inequality for an arbitrary $h \in \mathcal{M}_0(E)$ with $0 \leq h \leq f$. For such an h we have $0 \leq h \leq f \leq g$ and so it follows from the definition of the Lebesgue integral of g as a supremum that $\int_E h\, dx \leq \int_E g\, dx$.

□

Theorem (3)

Let $E \subset \mathbb{R}^q$ be a measurable set, $f : E \longrightarrow [0, \infty]$ be measurable function and A, B measurable and disjoint subsets of E. Then

$$\int_{A\cup B} f dx = \int_A f dx + \int_B f dx .$$

Proof.

It follows from $A \cap B = \emptyset$ that $\chi_{A\cup B} = \chi_A + \chi_B$. Hence

$$\int_{A\cup B} f dx = \int_E f\chi_{A\cup B} dx = \int_E f(\chi_A + \chi_B) dx = \int_E f\chi_A dx + \int_E f\chi_B dx = \int_A f dx + \int_B f dx .$$

□

A direct consequence is the following

Corollary (1)

Let $E \subset \mathbb{R}^q$ be a measurable set, $f : E \longrightarrow [0, \infty]$ be measurable function. If $E_0 \subset E$ has measure zero, then

$$\int_E f dx = \int_{E\setminus E_0} f dx .$$