Real Analysis MAA 6616
Lecture 15
Monotone Convergence Theorem
The General Lebesgue Integral

First recall that *liminf* of a sequence $\{a_n\}_n \subset \mathbb{R}$ is $\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right)$. If $\{b_n\}_n \subset \mathbb{R}$ converges and $b_n \leq a_n$ for all n, then $\lim_{n\to\infty} b_n \leq \liminf_{n\to\infty} a_n$.

Theorem (Fatou's Lemma 1)

Let $E \subset \mathbb{R}^q$ be a measurable set and $\{f_n\}_n$ a sequence of nonnegative measurable functions on E. If $f_n \longrightarrow f$ pointwise a.e. on E, then $\int_E f dx = \int_E \lim_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_E f_n dx.$

Proof.

There exists a set $S \subset E$ of measure 0 such that $f_n \longrightarrow f$ pointwise on $E_0 = E\backslash S$. The limit f is measurable is nonnegative.
To prove the theorem, it is enough to prove that $\int_E h dx \leq \liminf_{n\to\infty} \int_E f_n dx$, where h is an arbitrary bounded nonnegative, measurable function with finite support in E_0 (i.e. $h \in \mathcal{M}_0(E_0)$) such that $h \leq f$.
Let h be such function. Then there exists $M > 0$ such that $0 \leq h \leq M$. Let $F \subset E_0$ be the support of h so that $m(F) < \infty$ and $h = 0$ on $E_0\backslash F$. Consider the sequence of functions $\{h_n\}_n$ on E_0 given by $h_n = \min(h, f_n)$. Then h_n is nonnegative, measurable, with support in F, and $0 \leq h_n \leq f_n$. Note that $h_n \longrightarrow h$ on F. The sequence $\{h_n\}_n$ is uniformly bounded by M and converges to h. It follows then from the Bounded Convergence Theorem (Theorem 7 in Lecture 13) that $\lim_{n\to\infty} \int_F h_n dx = \int_F h dx$. On the other hand, since $0 \leq h_n \leq f_n$, it follows from the definition of $\int_F f_n$ that

$$\int_E h dx = \int_F h dx = \lim_{n\to\infty} \int_F h_n \leq \liminf_{n\to\infty} \int_F f_n \leq \liminf_{n\to\infty} \int_{E_0} f_n = \liminf_{n\to\infty} \int_E f_n dx$$

□

Recall that a sequence of functions $\{f_n\}_n$ is increasing (notation $\nearrow$) on a set E if $f_n(x) \leq f_{n+1}(x)$ for all $x \in E$. If $\{f_n\}$ is $\nearrow$ and converges to f we write $f_n \nearrow f$.

Theorem (Monotone Convergence Theorem 2)

Let $\{f_n\}_n$ be a sequence of measurable, nonnegative functions on a measurable set E. If $f_n \nearrow f$ a.e. on E, then $\lim_{n\to\infty} \int_E f_n dx = \int_E f dx.$

Proof.

Since $f_n \nearrow f$, then $f_n \leq f$ a.e. on E. It follows from the monotonicity of the integral of nonnegative functions that $\int_E f_n dx \leq \int_E f dx$ and so $\limsup_{n\to\infty} \int_E f_n dx \leq \int_E f dx$. On the other hand, it follows from Fatou's Lemma that $\int_E f dx \leq \liminf_{n\to\infty} \int_E f_n dx$. Therefore $\int_E f dx = \lim_{n\to\infty} \int_E f_n dx$. □

Corollary (1)

Let $\{u_n\}_n$ be a sequence of measurable, nonnegative functions on a measurable set E. If $\sum_{j=1}^{\infty} u_j = f$ *a.e. on E, then* $\int_E f dx = \int_E (\sum_{j=1}^{\infty} u_j) dx = \sum_{j=1}^{\infty} \int_E u_j dx.$

Proof.

Let $f_n = \sum_{j=1}^{n} u_j$, then $f_n \nearrow f$ a.e. on E. Apply Theorem 2 the sequence $\{f_n\}$. □

A measurable function $f : E \longrightarrow [0, \infty]$ is said to be integrable over E if $\int_E f dx < \infty$.

Proposition (1)

Let $E \subset \mathbb{R}^q$ be a measurable. If a measurable function $f : E \longrightarrow [0, \infty]$ is integrable over E, then it is finite a.e. on E.

Proof.

Let $\epsilon > 0$ and let $\lambda > 0$ such that $\int_E f dx < \epsilon\lambda$. It follows from Chebychev's inequality that $m(\{f > \lambda\}) \leq \frac{1}{\lambda}\int_E f dx < \epsilon$. Since $\{f = \infty\} \subset \{f > \lambda\}$, then $m(\{f = \infty\}) < \epsilon$. Therefore $m(\{f = \infty\}) = 0$. □

Theorem (Beppo Levi's Lemma 3)

Let $\{f_n\}_n$ be a sequence of measurable, nonnegative functions on a measurable set E. Suppose that $\{f_n\}_n$ is increasing and that the sequence of integrals $\left\{\int_E f_n dx\right\}_n$ is bounded. Then $f_n \nearrow f$ pointwise on E such that the limit f is finite a.e. on E, integrable over E, and

$$\lim_{n\to\infty} \int_E f_n dx = \int_E f dx.$$

Proof.

Since $\{f_n\}_n$ is increasing, then $f(x) = \lim_{n\to\infty} f(x)$ is well defined on $[0, \infty]$. It follows from the Monotone Convergence Theorem that $\lim_{n\to\infty} \int_E f_n dx = \int_E f dx$. Since the sequence of integrals $\left\{\int_E f_n dx\right\}_n$ is bounded, then $\int_E f dx$ is finite and so f is integrable over E and consequently finite a.e. on E. □

Let $E \subset \mathbb{R}^q$ be measurable and $f : E \longrightarrow \overline{\mathbb{R}}$. Recall that the positive and negative parts of f are the **nonnegative** functions f^+ and f^- defined on E by $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$. We have

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^- \text{ on } E$$

Note that since the space of measurable functions on E is a vector space, then f is measurable if and only if f^+ and f^- are measurable.

Lemma (1)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be measurable. Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.

Proof.

"$\Longleftarrow$" Suppose that $|f|$ is integrable. Since the nonnegative functions f^+ and f^- satisfy $0 \leq f^\pm \leq |f|$, the monotonicity of the integral of nonnegative functions implies that $\int_E f^\pm dx \leq \int_E |f|\, dx < \infty$. Therefore, f^+ and f^- are integrable.

"$\Longrightarrow$" Suppose that f^+ and f^- are integrable. It follows from the linearity of the integral of nonnegative functions that
$\int_E |f|\, dx = \int_E (f^+ + f^-) dx = \int_E f^+ dx + \int_E f^- dx < \infty$ $\square$

A measurable function $f : E \longrightarrow \overline{\mathbb{R}}$ is said to be integrable over E if $|f|$ is integrable over E. In this case its integral over E is

$$\int_E f dx = \int_E f^+ dx - \int_E f^- dx$$

Proposition (2)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be integrable. Then f is finite a.e. on E and if $S \subset E$ has measure 0, then
$$\int_E f dx = \int_{E\setminus S} f dx.$$

Proof.

Since $|f|$ is nonnegative and integrable, then $|f|$ (and so f) is finite a.e. on E. If $S \subset E$ has measure 0, then
$$\int_E f dx = \int_E f^+ dx - \int_E f^- dx = \int_{E\setminus S} f^+ dx - \int_{E\setminus S} f^- dx = \int_{E\setminus S} f dx$$
□

Proposition (3)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be measurable and let $g : E \longrightarrow [0, \infty]$ be integrable. If $|f| \leq g$ on E, then f is integrable and
$$\left| \int_E f dx \right| \leq \int_E |f| \, dx$$

Proof.

Since $|f| \leq g$ and g is integrable and so is $|f|$. It follows that
$$\left| \int_E f dx \right| = \left| \int_E (f^+ - f^-) dx \right| = \left| \int_E f^+ dx - \int_E f^- dx \right| \leq \int_E f^+ dx + \int_E f^- dx = \int_E (f^+ + f^-) dx = \int_E |f| \, dx$$
□

Note that it follows from Proposition 2 that the values of a function on a set of measure 0 do not affect the value of the integral. Now given integrable functions $f, g : E \longrightarrow \overline{\mathbb{R}}$, the set S of points where $f = \infty$ and $g = -\infty$ or $f = -\infty$ and $g = \infty$ has measure 0. In fact if $A = \{|f| < \infty\} \cap \{|g| < \infty\}$ then $E \setminus A$ has measure 0. If $f + g$ is integrable over A, then we set
$$\int_E (f + g) dx = \int_A (f + g) dx$$

Theorem (4)

Let $f, g : E \longrightarrow \overline{\mathbb{R}}$ be integrable. Then

1. *Linearity: For every $a, b \in \mathbb{R}$, the function $af + bg$ is integrable and*
$$\int_E (af + bg)dx = a\int_E fdx + b\int_E gdx$$
2. *Monotonicity: If $f \leq g$ on E, then $\int_E fdx \leq \int_E gdx$.*

Proof.

• Note that $|af| = |a|\,|f|$ and $|bg| = |b|\,|g|$ are integrable and so is $|af + bg| \leq |af| + |bg|$ and then $af + bg$.
To prove linearity, we start by proving that $\int_E (af)^{\pm} dx = a\int_E fdx$. Note also that if $a > 0$, then $(af)^{\pm} = af^{\pm}$ and if $a < 0$, then $(af)^{\pm} = -af^{\mp}$. For $a < 0$, the linearity of the integral for nonnegative functions gives
$\int_E (af)^{\pm} dx = \int_E (-a)f^{\mp} dx = -a\int_E f^{\mp} dx$. Hence
$$\int_E (af)dx = \int_E (af)^+ dx - \int_E (af)^- dx = -a\int_E f^- dx + a\int_E af^+ dx = a\left(\int_E (f^+ - f^-)dx\right) = a\int_E fdx.$$
Next, we prove that $\int_E (f+g)dx = \int_E fdx + \int_E gdx$. After removing the set of measure 0 (if necessary):
$S = \{|f| = \infty\} \cup \{|g| = \infty\}$ we can assume that both f and g are finite on E. We have to verify that
$$\int_E (f+g)^+ dx - \int_E (f+g)^- dx = \left[\int_E f^+ dx - \int_E f^- dx\right] + \left[\int_E g^+ dx - \int_E g^- dx\right]$$

□

Proof.

We have

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-).$$

This implies

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Since all six functions appearing in the above identity are nonnegative, then the linearity of the integral for nonnegative functions give

$$\int_E (f+g)+dx + \int_E f^- dx + \int_E g^- dx = \int_E (f+g) - dx + \int_E f^+ dx + \int_E g^+ dx.$$

From this we deduce

$$\int_E (f+g)dx = \int_E f dx + \int_E g dx$$

• Suppose that f and g are finite on E and $g \geq f$. Let $h = g - f$. Then h is a nonnegative integrable function on E. It follows from part 1 that

$$\int_E f dx = \int_E (g-h)dx = \int_E g dx - \int_E h dx \leq \int_E g dx$$

□

Theorem (5)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be integrable and let A and B be disjoint measurable subsets of E. Then

$$\int_{A\cup B} fdx = \int_A fdx + \int_B fdx$$

Proof.

First note that if $F \subset E$ is measurable, then $|f\chi_F|$ $(\leq |f|)$ is integrable and so is $f\chi_F$. moreover $\int_F fdx = \int_E f\chi_F dx$.
Since A and B are disjoint, then $\chi_{A\cup B} = \chi_A + \chi_B$, then

$$\int_{A\cup B} fdx = \int_E \left(f\chi_A + f\chi_B\right) dx = \int_E f\chi_A dx + \int_E f\chi_B dx = \int_A fdx + \int_B fdx .$$

□