Real Analysis MAA 6616
Lecture 16
Convergence of Lebesgue Integrals

Theorem (1. Lebesgue Dominated Convergence Theorem)

Let $\{f_n\}_n$ be a sequence of measurable functions on a set E such that $f_n \longrightarrow f$ a.e. on E. Suppose that there exists a sequence $\{g_n\}_n$ of nonnegative integrable functions on E such that

- $g_n \longrightarrow g$ *a.e. on E with* $\lim_{n\to\infty} \int_E g_n dx = \int_E g dx < \infty$;
- $|f_n| \leq g_n$ *for all* $n \in \mathbb{N}$

Then f is integrable and $\lim_{n\to\infty} \int_E f_n dx = \int_E f dx.$

Proof.

First note that if $\{a_n\}_n$ and $\{b_n\}_n$ are sequences of real numbers such that $a_n \longrightarrow a$, then $\liminf_{n\to\infty}(a_n + b_n) = a + \liminf_{n\to\infty} b_n$ and $\liminf_{n\to\infty}(a_n - b_n) = a - \limsup_{n\to\infty} b_n$.

Since $|f_n| \leq g_n$ and g_n integrable, then $|f_n|$ is integrable. It follows from Fatou's Lemma that

$$\int_E |f|\, dx \leq \liminf_{n\to\infty} \int_E |f_n|\, dx \leq \liminf_{n\to\infty} \int_E |g_n|\, dx = \int_E g dx < \infty$$

Hence, f is integrable. We are left to show that $\int_E f dx = \lim_{n\to\infty} \int_E f_n dx$.

For this we use again Fatou's Lemma and the linearity of the integral

$$\int_E f dx = \int_E g dx - \int_E (g - f) dx \geq \int_E g dx - \liminf_{n\to\infty} \int_E (g_n - f_n) dx = \int_E g dx - \int_E g dx + \limsup_{n\to\infty} \int_E f_n dx$$

Also

$$\int_E f dx = \int_E (g + f) dx - \int_E g dx \leq \liminf_{n\to\infty} \int_E (g_n + f_n) dx - \int_E g dx = \int_E g dx + \liminf_{n\to\infty} \int_E f_n dx - \int_E g dx$$

This means $\limsup_{n\to\infty} \int_E f_n dx \leq \int_E f dx \leq \liminf_{n\to\infty} \int_E f_n dx$. This proves $\int_E f dx = \lim_{n\to\infty} \int_E f_n dx$. □

Theorem (2)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be an integrable function. If $\{E_n\}_{n=1}^{\infty}$ is a disjoint collection of measurable subsets of E, then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f dx = \sum_{n=1}^{\infty} \int_{E_n} f dx \, .$$

Proof.

Let $F = \bigcup_{n=1}^{\infty} E_n$. For each $n \in \mathbb{N}$, let $A_n = \bigcup_{j=1}^{n} E_j$ and let $f_n = f\chi_{A_n}$. Then f_n is measurable on E and $|f_n| \leq |f|$ on E and $f_n \longrightarrow f$ pointwise on F. It follows from Lebesgue Dominated Convergence Theorem (with $g_n = |f|$ for all n) that $\int_F f dx = \lim_{n\to\infty} \int_F f_n dx.$

Since the En's are disjoint, it follows from the (finite) additivity of the integral that

$$\int_F f_n dx = \int_F f\chi_{A_n} dx = \sum_{j=1}^{n} \int_{E_j} f dx \, .$$

Hence

$$\int_{\bigcup_{n=1}^{\infty} E_n} f dx = \lim_{n\to\infty} \int_F f_n dx = \lim_{n\to\infty} \left[\sum_{j=1}^{n} \int_{E_j} f dx \right] = \sum_{n=1}^{\infty} \int_{E_n} f dx \, .$$

□

Theorem (3)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be an integrable function.

1. *If $\{E_n\}_{n=1}^{\infty}$ is an ascending collection of measurable subsets of E, then*
$$\int_{\bigcup_{n=1}^{\infty} E_n} fdx = \lim_{n\to\infty} \int_{E_n} fdx$$
2. *If $\{E_n\}_{n=1}^{\infty}$ is a descending collection of measurable subsets of E, then*
$$\int_{\bigcap_{n=1}^{\infty} E_n} fdx = \lim_{n\to\infty} \int_{E_n} fdx$$

Lemma (1)

Let $E \subset \mathbb{R}^q$ be measurable and with finite measure and let $\delta > 0$. Then E can be written as a disjoin union of a finite collection of measurable sets each of which has measure $< \delta$: That is, there exist disjoint measurable sets $A_1, \cdots, A_N$ such that $E = A_1 \cup \cdots \cup A_N$ and for every $j = 1, \cdots, N$, we have $m(A_j) < \delta$.

Proof.

For $n \in \mathbb{N}$ consider the cube C_n in $\mathbb{R}^q$ given by $C_n = [-n,\ n]^q$. Let $E_n = E \cap C_n$. Then $\{E_n\}$ is an ascending collection of measurable sets and $E = \bigcup_n E_n$. Then $m(E) = \lim_{n\to\infty} m(E_n)$. Let $F_n = E \backslash E_n$ so that $E = E_n \cup F_n$. Since E has finite measure, then $\lim_{n\to\infty} m(F_n) = 0$. Let $n_0 \in \mathbb{N}$ such that $m(F_{n_0}) < \delta$. We are left to prove the Lemma for E_{n_0}.

Let $M \in \mathbb{N}$ be such that $\left(\frac{2n_0}{M}\right)^q < \delta$. Divide $[-n_0,\ n_0]$ into M intervals by the points $x_j = -n_0 + j(2n_0/M)$, with $j = 0, \cdots, M$. Let $I_j = [x_j,\ x_{j+1})$ so that $\ell(I_j) = 2n_0/M$. For a multi index $\alpha = (j_1, \cdots, j_q) \in H$ with $H = \{0, \cdots, M-1\}^q \subset \mathbb{Z}^q$, define the cube $D_\alpha \in \mathbb{R}^q$ given by $D_\alpha = I_{j_1} \times \cdots \times I_{j_q}$. Then $m(D_\alpha) = (2n_0/M)^q < \delta$. Note that $C_{n_0} = \bigcup_{\alpha\in H} D_\alpha$. For every $\alpha \in H$ let $A_\alpha = D_\alpha \cap E$. The collection of $\{A_\alpha\}_{\alpha\in H}$ is disjoint and $E_{n_0} = \bigcup_{\alpha\in H} A_\alpha$ and $m(A_\alpha) \le m(D_\alpha) < \delta$.

Proposition (1)

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be a measurable function and $m(E) < \infty$. Then f is integrable over E if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable set $A \subset E$ with $m(A) < \delta$, we have $\int_A |f|\, dx < \epsilon$.

Proof.

Since $f = f^+ - f^+$, and $|f| = f^+ + f^-$, if the theorem holds for nonnegative functions then it holds for general functions. So we can assume that f is nonnegative.

"$\Longrightarrow$" Suppose f is integrable. Let $\epsilon > 0$. By definition of the integral of nonnegative functions, there exists a measurable bounded function f_ϵ with finite support such that $0 \leq f_\epsilon \leq f$ on E and $\int_E f_\epsilon dx \leq \int_E f dx \leq \int_E f_\epsilon dx + (\epsilon/2)$. It follows from the linearity of the integral that for any measurable set $A \subset E$ we have

$$\int_A f dx - \int_A f_\epsilon dx = \int_A (f - f_\epsilon) dx \leq \int_E (f - f_\epsilon) dx = \int_E f dx - \int_E f_\epsilon dx \leq \frac{\epsilon}{2}$$

Since f_ϵ is bounded, let $M > 0$ such that $0 \leq f_\epsilon < M$ on E. It follows that

$$0 \leq \int_A f dx \leq \int_A f_\epsilon dx + \frac{\epsilon}{2} \leq Mm(A) + \frac{\epsilon}{2}.$$

For $\delta = \frac{\epsilon}{2M}$ we get $0 \leq \int_A f dx \leq \epsilon$.

"$\Longleftarrow$" Suppose that for each ϵ there exists $\delta > 0$ such that for every measurable set $A \subset E$ with $m(A) < \delta$, we have $\int_A |f|\, dx < \epsilon$. Select $\epsilon = 1$ and the corresponding $\delta = \delta_0$. It follows from Lemma 1 that there exists a finite collection of disjoints measurable sets $E_1, \cdots, E_N$ such that $E = E_1 \cup \cdots \cup E_N$ and $m(E_j) \leq \delta_0$ for every $j = 1, \cdots, N$. It follows that $\sum_{j=1}^{N} \int_{E_j} f dx < N$. Now if h is an arbitrary nonnegative bounded function with finite support such that $0 \leq h \leq f$ on E, then $\int_E h dx < N$. This implies that f is integrable over E and $\int_E f dx < N$. □

Remark (1)

The implication "$\Longrightarrow$" is still valid even without the assumption that E has finite measure.

A collection $\mathcal{F}$ of measurable functions on a set $E \subset \mathbb{R}^q$ is said to be uniformly integrable over E if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $f \in \mathcal{F}$, we have $\int_A |f|\, dx \leq \epsilon$ whenever $A \subset E$ has measure $m(A) \leq \delta$.

Note that a finite family $\mathcal{F} = \{f_j\}_{j=1}^{n}$ of integrable functions on E is always uniformly integrable. This follows from Proposition 1.

Proposition (2)

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on a set $E \subset \mathbb{R}^q$ with finite measure. Suppose that this sequence is uniformly integrable and $f_n \longrightarrow f$ pointwise a.e. on E. Then the limit f is integrable over E.

Proof.

We need to prove that $\int_E |f|\, dx < \infty$. First note that since $f_n \longrightarrow f$ a.e. on E, then it follows from Fatou's Lemma that $\int_E |f|\, dx \leq \liminf_{n\to\infty} \int_E |f_n|\, dx$.

Now we use the uniform integrability of the sequence $\{f_n\}_n$ with $\epsilon = 1$ to find $\delta > 0$ such that $\int_A |f_n|\, dx < 1$ for every measurable set $A \subset E$ with $m(A) < \delta$ and for every n. Since $m(E) < \infty$, then we can find a finite collection of disjoint measurable sets $\{A_1, \cdots, A_N\}$ such that $E = A_1 \cup \cdots \cup A_N$ and $m(A_j) < \delta$ for $j = 1, \cdots, N$.

For any $n \in \mathbb{N}$ we have $\int_E |f_n|\, dx = \sum_{j=1}^{N} \int_{A_j} |f_n|\, dx < N$. Therefore $\int_E |f|\, dx \leq N$ and f is integrable. □

Theorem (4. Vitali Convergence Theorem)

Let $E \subset \mathbb{R}^q$ be measurable with finite measure. Suppose that the sequence of functions $\{f_n\}_n$ is uniformly integrable over E and that $f_n \longrightarrow f$ pointwise a.e. on E. Then f is integrable over E and $\int_E f dx = \lim_{n\to\infty} \int_E f_n dx$.

Proof.

We already know from Proposition 2 that the limit function f is integrable over E. It remains to prove $\int_E f dx = \lim_{n\to\infty} \int_E f_n dx$. We can find a set $S \subset E$ such that $m(S) = 0$, $f_n \longrightarrow f$ pointwise on $F = E \backslash S$ and $|f| < \infty$ on F (because f is finite a.e. on E as an integrable function). It suffices therefore to establish the result when E is replaced by F. If $A \subset F$ is any measurable set, then it follows from the linearity and monotonicity of the integral that

$$\left|\int_F f dx - \int_F f_n dx\right| \leq \int_F |f - f_n|\, dx \leq \int_{F\backslash A} |f - f_n|\, dx + \int_A |f - f_n|\, dx \leq \int_{F\backslash A} |f - f_n|\, dx + \int_A |f|\, dx + \int_A |f_n|\, dx$$

Now let $\epsilon > 0$. By using the uniform integrability of the sequence $\{f_n\}$, we can find $\delta > 0$ such that $\int_A |f_n|\, dx \leq (\epsilon/3)$ whenever $A \subset F$ has measure $m(A) < \delta$. We also have $\int_A |f|\, dx \leq (\epsilon/3)$ (Fatou's Lemma). Since $m(F) = m(E) < \infty$, then it follows from Egorov's Theorem that we can find a set $A_0 \subset F$ with $m(A_0) < \delta$ such that $f_n \longrightarrow f$ uniformly on $F \backslash A_0$. Hence, there exists $N \in \mathbb{N}$ such that $|f - f_n| \leq (\epsilon/(3m(E)))$ for every $n > N$. Finally, using the set A_0, we get

$$\left|\int_F f dx - \int_F f_n dx\right| \leq \int_{F\backslash A_0} |f - f_n|\, dx + \int_{A_0} |f|\, dx + \int_{A_0} |f_n|\, dx \leq \frac{\epsilon}{3m(E)}\, m(F\backslash A_0) + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\,.$$

This implies that $\int_E |f|\, dx = \lim_{n\to\infty} \int_E |f_n|\, dx.$ □

The following theorem justifies the importance of uniform integrability in the passage to the limit under the integral sign.

Theorem (5)

Let $E \subset \mathbb{R}^q$ with finite measure. Suppose that $\{h_n\}_n$ is a sequence of nonnegative integrable functions on E such that $h_n \longrightarrow 0$ pointwise a.e. on E. Then $\lim_{n\to\infty} \int_E h_n dx = 0$ if and only if the sequence $\{h_n\}_n$ is uniformly integrable over E

Proof.

"$\Longleftarrow$" This is a consequence of Theorem 4.

"$\Longrightarrow$" Suppose that $\lim_{n\to\infty} \int_E h_n dx = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $0 \leq \int_E h_n dx < \epsilon$ for all $n > N$.

Since $h_n \geq 0$, then we also have $\int_A h_n dx < \epsilon$ for any measurable set $A \subset E$.

The finite family $\{h_1, \cdots, h_N\}$ is uniformly integrable. Indeed, for each $j = 1, \cdots, N$ there exists $\delta_j > 0$ such that $\int_A h_j dx \leq \epsilon$ whenever $m(A) < \delta_j$. Let $\delta = \min(\delta_1, \cdots, \delta_N)$. For $A \subset E$ with $m(A) < \delta$ we have $\int_A h_n dx < \epsilon$ for any n. □