Real Analysis MAA 6616
Lecture 17
Repeated Integration: Fubini's Theorem

It follows from the definition of the measure that if I and J are boxes in Euclidean spaces, then $I \times J$ is a box and $v(I \times J) = v(I) \times v(J)$. This property extend to products of measurable sets. We will show that the Lebesgue measure of a product of two sets $E \times F$ is the product of the measures of E and F.

Proposition (1)

Let $Z \subset \mathbb{R}^r$ and $F \subset \mathbb{R}^s$ be measurable sets with $m(Z) = 0$. Then $Z \times F \subset \mathbb{R}^{r+s}$ is measurable and $m(Z \times F) = 0$.

Proof.

First consider the case $m(F) < \infty$. Let $p, q \in \mathbb{N}$. Then there exist open sets $U_p \subset \mathbb{R}^r$, $V_q \subset \mathbb{R}^s$ such that $Z \subset U_p$, $F \subset V_q$, $m(U_p) < (1/p)$, and $m(V_q) < m(F) + (1/q)$.

There exist countable collections of nonoverlapping boxes $\{I_{i,p}\}_i$ and $\{J_{j,q}\}_j$ such that $U_p = \bigcup_{i=1}^{\infty} I_{i,p}$ and $V_q = \bigcup_{j=1}^{\infty} J_{j,q}$.

We have then $U_p \times V_q = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,p} \times J_{j,q}$. Therefore

$$m(U_p \times V_q) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} m(I_{i,p}) \cdot m(J_{j,q}) = \sum_{i=1}^{\infty} m(I_{i,p}) \cdot \sum_{j=1}^{\infty} m(J_{j,q}) = m(U_p) \cdot m(V_q) \leq \frac{1}{p} \left(m(F) + \frac{1}{q} \right)$$

Since $Z \times F \subset U_p \times V_q$ and p, q are arbitrary, it follows that $m(Z \times F) = 0$.

If $m(F) = \infty$. For $n \in \mathbb{N}$, let $F_n = F \cap B_n(0)$, where $B_n(0)$ is the ball with center 0 and radius n in $\mathbb{R}^s$. We have $Z \times F = \bigcup_{n=1}^{\infty} Z \times F_n$. Since $m(F_n) < \infty$, then (previous case) $m(Z \times F_n) = 0$ for all n, therefore $m(Z \times F) = 0$. □

Remark (1)

The measurability of a product $A \times B$ of two sets does not imply the measurability of each set. For example. Let $A = Z$ be set of measure 0 and B be any bounded **nonmeasurable** set. Then $A \times B$ is measurable with measure 0.

Proposition (2)

Let $E \subset \mathbb{R}^r$ and $F \subset \mathbb{R}^s$ be measurable sets. Then $E \times F$ is a measurable set in $\mathbb{R}^{r+s}$ and $m(E \times F) = m(E)m(F)$ if $m(E) \neq 0$ and $m(F) \neq 0$. If one of the sets has measure 0, then $m(E \times F) = 0$.

Proof.

Since the case $m(E) = 0$ or $m(F) = 0$ is dealt with in Proposition 1, we assume that none of the sets has measure 0. There exist G_δ sets G^1 and G^2 such that $E \subset G^1, F \subset G^2$ and $m(G^1 \backslash E) = 0, m(G^2 \backslash F) = 0$. Since the G_δ set $G^1 \times G^2$ contains $E \times F$, then $E \times F$ would be measurable if $m\left((G^1 \times G^2) \backslash (E \times F)\right) = 0$.

We have $G^1 \times G^2 \backslash E \times F = (G^1 \backslash E) \times G^2 \cup G^1 \times (G^2 \backslash F)$. Since $G^1 \backslash E$ and $G^2 \backslash F$ have measure 0, then $(G^1 \backslash E) \times G^2$ and $G^1 \times (G^2 \backslash F)$ have measure 0 and so does $(G^1 \times G^2) \backslash (E \times F)$. Hence $E \times F$ is measurable and $m(E \times F) = m(G^1 \times G^2)$.

We can write $G^1 = \bigcap_{p=1}^{\infty} U_p$ and $G^2 = \bigcap_{q=1}^{\infty} V_q$ with U_p, V_q open in $\mathbb{R}^r$ and$\mathbb{R}^s$, respectively. For each p and q, there exist countable collections of non overlapping boxes $\{I_{i,p}\}_i$ and $\{J_{j,q}\}_j$ such that $U_p = \bigcup_{i=1}^{\infty} I_{i,p}$ and $V_q = \bigcup_{j=1}^{\infty} J_{j,q}$. As in the proof of Proposition 1, we have $m(U_p \times V_q) = m(U_p)m(V_q)$. It follows that $G^1 \times G^2 = \bigcap_{p,q=1}^{\infty} U_p \times V_q$ has measure

$$m(G^1 \times G^2) = \lim_{p \to \infty} m(U_p) \cdot \lim_{q \to \infty} m(V_q) = m(G^1)m(G^2) = m(E) \times m(F).$$

□

Let $S \subset \mathbb{R}^d$ be a measurable set. We denote by $\mathcal{L}(S)$ the vector space of $\overline{R}$-valued functions that are Lebesgue integrable on S:

$$\mathcal{L}(S) = \left\{ f : S \longrightarrow \overline{\mathbb{R}} : \int_S |f|\, dx < \infty \right\}$$

Let $E \subset \mathbb{R}^r$ and $F \subset \mathbb{R}^s$ be measurable sets. The points in $\mathbb{R}^r$ will be denoted by $x = (x_1, \cdots, x_r)$ and the points in $\mathbb{R}^s$ will be denoted by $y = (y_1, \cdots, y_s)$. Let $f : E \times F \longrightarrow \overline{\mathbb{R}}$ be a measurable function. The integral of f over $E \times F$ will be denoted by $\displaystyle\int_{E\times F} f(x,y)dxdy$. For a given $x \in E$, we denote by $\displaystyle\int_F f(x,y)dy$ the integral of $f(x,\cdot) : F \longrightarrow \overline{\mathbb{R}}$ and for a given $y \in F$, we denote by $\displaystyle\int_E f(x,y)dx$ the integral of $f(\cdot, y) : E \longrightarrow \overline{\mathbb{R}}$.

To the function $f : E \times F \longrightarrow \overline{\mathbb{R}}$, we can associate the following three integrals: The double integral

$$\int_{E\times F} f(x,y)dxdy\,,$$

and the iterated integrals

$$\int_E \left[\int_F f(x,y)dy \right] dx \;\; \text{and} \;\; \int_F \left[\int_E f(x,y)dx \right] dy\,.$$

Fubini Theorem gives sufficient conditions for the three integrals to be equal.

Remark (2)

If a function $f(x,y)$ is measurable on a set $A \times B$, then this alone does not imply that $f(x,\cdot)$ is measurable for all x in A. For example Let $A = Z$ be a set with measure 0, $B = N$ a nonmeasurable set. Then $Z \times N$ is measurable with measure 0. The function $f(x,y) = \chi_{Z\times N}$ is measurable. However for $x \in Z$, the function $f(x,\cdot) = \chi_N$ is not measurable.

Theorem (1. Fubini's Theorem)

Let $E \subset \mathbb{R}^r$ and $F \subset \mathbb{R}^s$ be measurable sets and let $f : E \times F \longrightarrow \overline{\mathbb{R}}$. Suppose that $f \in \mathcal{L}(E \times F)$. Then

1. *For almost all $x \in E$, $f(x, \cdot)$ is a measurable function of $y \in F$ and $f(x, \cdot) \in \mathcal{L}(F)$.*
2. *The function $A : E \longrightarrow \mathbb{R}$ defined for a.e. $x \in E$ by*

$$A(x) = \int_F f(x, y)dy \quad \text{is in } \mathcal{L}(E)\,.$$

3. *For almost all $y \in F$, $f(\cdot, y)$ is a measurable function of $x \in E$ and $f(\cdot, y) \in \mathcal{L}(E)$.*
4. *The function $B : F \longrightarrow \mathbb{R}$ defined for a.e. $y \in F$ by*

$$B(y) = \int_E f(x, y)dx \quad \text{is in } \mathcal{L}(F).$$

5. *Moreover, we have*

$$\int_{E\times F} f(x, y)dxdy = \int_E \left[\int_F f(x, y)dy\right] dx = \int_F \left[\int_E f(x, y)dx\right] dy\,.$$

Fubini's Theorem will be proved in several steps going from simple to more general situations.

Remark (3)

- Reduction to case $E = \mathbb{R}^r$ and $F = \mathbb{R}^s$. Since $f \in \mathcal{L}(E \times F)$, then we can extend the function f by defining it to be 0 on $\mathbb{R}^r \times \mathbb{R}^s \backslash E \times F$. To obtain a function $\tilde{f} \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$ and prove the theorem for $\tilde{f}$.
- A partly open interval (or box or rectangle) in $\mathbb{R}^d$ is $I = I_1 \times \cdots \times I_d$ where each I_j is an interval in $\mathbb{R}$ of the form $[a_j,\ b_j)$. The boundary of ∂I of I is the set of points $x = (x_1, \cdots, x_d)$ such that for some j, $x_j = a_j$ or $x_j = b_j$.
- For $I = [a,\ b)$ and J an interval in $\mathbb{R}^s$, we have $\partial(I \times J) = (\{a\} \times J) \cup (\{b\} \times J) \cup (I \times \partial J)$. Since $m(\partial J) = 0$, then $m\left(\{y \in \mathbb{R}^s :\ (x, y) \in \partial(I \times J)\}\right) = 0$ for all x in the interior of I.
- For any given open set $U \subset \mathbb{R}^d$, there exists a countable collection of disjoint party open intervals $\{I_n\}_n$ in $\mathbb{R}^d$ such that $U = \bigcup_{n=1}^{\infty} I_n$.

Denote by Fub $(\mathbb{R}^r \times \mathbb{R}^s)$ the subset of integrable functions in $\mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$ that satisfy the five properties listed in Fubini's Theorem. Our aim is to prove that Fub $(\mathbb{R}^r \times \mathbb{R}^s) = \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$.

Lemma (1)

Fub $(\mathbb{R}^r \times \mathbb{R}^s)$ *is a vector space. That is if* $f, g \in$ Fub $(\mathbb{R}^r \times \mathbb{R}^s)$ *and* $a, b \in \mathbb{R}$, *then* $af + bg \in$ Fub $(\mathbb{R}^r \times \mathbb{R}^s)$.

This follows from the linearity of the integral.

Lemma (2)

Let $A \subset \mathbb{R}^r$ and $B \subset \mathbb{R}^s$ be measurable with finite measures, then $\chi_{A\times B} \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof.

Let $f(x,y) = \chi_{A\times B}(x,y) = \chi_A(x)\chi_B(y)$. Hence $f(x,\cdot) = 0$ if $x \notin A$ and $f(x,\cdot) = \chi_B$ if $x \in A$. Since χ_B is measurable, then $f(x,\cdot)$ is measurable and $f(x,\cdot) \in \mathcal{L}(\mathbb{R}^s)$ for every $x \in \mathbb{R}^r$. We have
$\int_{\mathbb{R}^s} f(x,y)dy = \chi_A(x)m(B) \in \mathcal{L}(\mathbb{R}^r)$.

Similarly $f(\cdot,y) \in \mathcal{L}(\mathbb{R}^r)$ for every $y \in \mathbb{R}^s$ and $\int_{\mathbb{R}^r} f(x,y)dx = \chi_B(y)m(A) \in \mathcal{L}(\mathbb{R}^s)$.

Finally $\int_{\mathbb{R}^r\times\mathbb{R}^s} \chi_{A\times B}dxdy = m(A\times B) = m(A)m(B)$ and

$$\int_{\mathbb{R}^r}\left[\int_{\mathbb{R}^s}\chi_{A\times B}dy\right]dx = \int_{\mathbb{R}^r} m(B)\chi_A\,dx = m(A)m(B) = \int_{\mathbb{R}^s}\left[\int_{\mathbb{R}^r}\chi_{A\times B}dx\right]dy$$

□

Lemma (3)

Let $\{f_n\}_n \subset$ Fub $(\mathbb{R}^r \times \mathbb{R}^s)$ be a sequence of nonnegative functions. Suppose that $f_n \nearrow f$ (or $f_n \searrow f$) with $f \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$, then $f \in$ Fub $(\mathbb{R}^r \times \mathbb{R}^s)$

Proof.

Note that since $\{f_n\} \subset \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$ (and so $f_n \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$) and since $f_n \nearrow f$ with $f \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$, then It follows from the MCT (Monotone Convergence Theorem) that $\lim_{n\to\infty}\int_{\mathbb{R}^{r+s}} f_n(x,y)dxdy = \int_{\mathbb{R}^{r+s}} f(x,y)dxdy$.

For every $n \in \mathbb{N}$ there exists a set $Z_n \subset \mathbb{R}^r$ with $m(Z_n) = 0$ such that $f_n(x,\cdot) \in \mathcal{L}(\mathbb{R}^s)$. Let $Z = \bigcup_{n=1}^{\infty} Z_n$. Then $m(Z) = 0$ and $f_n(x,\cdot) \nearrow f(x,\cdot)$ for all $x \notin Z$. Let $g_n(x) = \int_{R^s} f_n(x,y)dy$ and $g(x) = \int_{R^s} f(x,y)dy$. The MCT implies that $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \notin Z$. Since $f_n \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$, then $g_n \in \mathcal{L}(\mathbb{R}^r)$ for $x \notin Z$ and $g_n \nearrow g$. The MCT once again gives $\lim_{n\to\infty}\int_{\mathbb{R}^r} g_n(x)dx = \int_{\mathbb{R}^r} g(x)dx$ for $x \notin Z$.

□

Proof.

(CONTINUED) This means that for $x \notin Z$ we have

$$\int_{\mathbb{R}^{r+s}} f(x,y)dxdy = \lim_{n\to\infty} \int_{\mathbb{R}^{r+s}} f_n(x,y)dxdy = \lim_{n\to\infty} \int_{\mathbb{R}^r} g_n(x)dx = \int_{\mathbb{R}^r} g(x)dx = \int_{\mathbb{R}^r} \left[\int_{\mathbb{R}^s} f(x,y)dy\right] dx$$

A similar argument gives the equality with the third integral. Therefore $f \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$. □

Lemma (4)

Let I and J be intervals in $\mathbb{R}^r$ and R^s and let $E \subset \partial(I \times J)$. Then $\chi_E \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof.

Since $m(\partial(I \times J)) = 0$, the $m(E) = 0$, the function $\chi_E \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$ and $\int_{\mathbb{R}^r \times \mathbb{R}^s} \chi_E dxdy = m(E) = 0$. We have $m[\{y \in J : (x,y) \in \partial(I \times J)\}] = 0$ for almost every $x \in I$ (see Remark 3). Therefore, $\chi_E(x,\cdot)$ has support with measure 0 for a.e. $x \in I$, hence $g(x) = \int_{\mathbb{R}^s} \chi_E(x,y)dy = 0$ a.e. $x \in I$. Hence

$0 = \int_{\mathbb{R}^r \times \mathbb{R}^s} \chi_E dxdy = \int_{\mathbb{R}^r} g(x)dx = \int_{\mathbb{R}^r}\left[\int_{\mathbb{R}^s} \chi_E(x,y)dy\right] dx$. A similar relation holds by interchanging the roles of x and y. Thus $\chi_E \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$. □

Lemma (5)

Let $U \subset \mathbb{R}^r \times \mathbb{R}^s$ be an open set with $m(U) < \infty$. Then $\chi_U \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof.

There exists a countable collection $\{I_p \times J_p\}_{p=1}^{\infty}$ of disjoint, partly open intervals in $\mathbb{R}^r \times \mathbb{R}^s$ such that $U = \bigcup_{p=1}^{\infty} I_p \times J_p$.

For $n \in \mathbb{N}$ let $U_n = \bigcup_{p=1}^{n} I_p \times J_p$. The collection of measurable sets $\{U_n\}_n$ is ascending, $U = \bigcup_{n=1}^{\infty} U_n$ and $\chi_{U_n} \nearrow \chi_U$.

Since $\chi_{U_n} = \sum_{p=1}^{n} \chi_{I_p \times J_p}$ and $\chi_{I_p \times J_p} \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$ (Lemma 2), then $\chi_{U_n} \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$ (Lemma 1).

Consequently $\chi_U \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$ (Lemma 3). □

Lemma (6)

Let $G \subset \mathbb{R}^r \times \mathbb{R}^s$ be a G_δ set: $G = \bigcap_{p=1}^{\infty} U_p$, where $U_p \subset \mathbb{R}^r \times \mathbb{R}^s$ open and $m(U_1) < \infty$. Then $\chi_G \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof.

We can write G as $G = \bigcap_{n=1}^{\infty} V_n$ where where $V_n = \bigcap_{p=1}^{n} U_p$ so that $\{V_n\}_n$ is a descending collection of open sets and $G = \bigcap_{n=1}^{\infty} V_n$. We have $\chi_{V_n} \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$ (Lemma 5) and $\chi_{V_n} \searrow \chi_G$. Therefore $\chi_G \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$. □

Lemma (7)

Let $Z \subset \mathbb{R}^{r+s}$ with $m(Z) = 0$. Then $\chi_Z \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$. Moreover $m[\{y \in \mathbb{R}^s : (x,y) \in Z\}] = 0$ a.e. $x \in \mathbb{R}^r$ and $m[\{x \in \mathbb{R}^r : (x,y) \in Z\}] = 0$ a.e. $y \in \mathbb{R}^s$

Proof.

We can find a G_δ set G such that $Z \subset G$ and $m(G) = m(Z) = 0$. Without loss of generality, we can assume that $G = \bigcap_{n=1}^{\infty} U_n$ with U_n open and $m(U_1) < \infty$. Hence (Lemma 6), $\chi_G \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$ and $0 = \int_{\mathbb{R}^r \times \mathbb{R}^s} \chi_G \, dxdy = \int_{\mathbb{R}^r} \left[\int_{\mathbb{R}^s} \chi_G(x,y) dy \right] dx$. It follows that $g(x) = \int_{\mathbb{R}^s} \chi_G(x,y) dy = 0$ a.e. $x \in \mathbb{R}^r$. Equivalently $m[\{y \in \mathbb{R}^s : (x,y) \in G\}] = 0$ a.e. $x \in \mathbb{R}^r$. Consequently $m[\{y \in \mathbb{R}^s : (x,y) \in Z\}] = 0$ a.e. $x \in \mathbb{R}^r$, since $Z \subset G$. It follows at once $\chi_Z(x, \cdot) \in \mathcal{L}(\mathbb{R}^s)$ and $\int_{\mathbb{R}^s} \chi_Z(x,y) dy = 0$ a.e. $x \in \mathbb{R}^r$ and $0 = \int_{\mathbb{R}^r \times \mathbb{R}^s} \chi_Z \, dxdy = \int_{\mathbb{R}^r} \left[\int_{\mathbb{R}^s} \chi_Z(x,y) dy \right] dx$. A similar argument holds when the roles of x and y are interchanged. This shows that $\chi_Z \in \text{Fub}(\mathbb{R}^r \times \mathbb{R}^s)$. □

Lemma (8)

Let $E \subset \mathbb{R}^r \times \mathbb{R}^s$ be measurable with $m(E) < \infty$. Then $\chi_E \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof.

There exists a G_δ set G such that $E \subset G$ and $m(G \backslash E) = 0$. Let $Z = G \backslash E$. We have $E = G \backslash Z$ and $\chi_E = \chi_G - \chi_Z$. It follows from Lemmas 6 and 7 that $\chi_G \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$ and $\chi_Z \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$. Therefore $\chi_E \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$ by Lemma 1. □

Proof.

Fubini's Theorem: Let $f \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$. We can write $f = f^+ - f^-$. Both nonnegative functions f^+ and f^- are in $\mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$. Since $\text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$ is a vector space (Lemma 1), it is enough to prove $f^\pm \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$. Without loss of generality we can assume f is nonnegative. Since f is measurable, we can find a sequence of nonnegative simple functions $\{f_n\}_n$ with $f_n \in \mathcal{L}(\mathbb{R}^r \times \mathbb{R}^s)$ and $f_n \nearrow f$.

For each $n \in \mathbb{N}$, we can find sets $E_n^1, \cdots, E_n^{p(n)}$ with finite measures in $\mathbb{R}^r \times \mathbb{R}^s$ and real numbers $c_n^1, \cdots, c_n^{p(n)}$ such that $f_n = \sum_{j=1}^{p(n)} c_j \chi_{E_n^j}$. It follows then from Lemmas 1 and 8 that $f_n \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$. Finally, since $f_n \nearrow f$, Lemma 3 implies that $f \in \text{Fub}\,(\mathbb{R}^r \times \mathbb{R}^s)$. □

Theorem (2)

Let $A \subset \mathbb{R}^{r+s}$ be measurable and $f : A \longrightarrow \overline{\mathbb{R}}$ be a measurable function. For $x \in \mathbb{R}^r$, let $A_x = \{y \in \mathbb{R}^s : (x, y) \in A\}$. Then

- *For a.e. $x \in \mathbb{R}^r$, A_x is a measurable subset of $\mathbb{R}^s$ and $f(x, \cdot)$ is a measurable function of $y \in A_x$.*
- *If $f \in \mathcal{L}(A)$, then for a.e. $x \in \mathbb{R}^r$, $f(x, \cdot) \in \mathcal{L}(A_x)$. Moreover, the function $g(x) = \displaystyle\int_{A_x} f(x, y)dy$ is in $\mathcal{L}(\mathbb{R}^r)$ and*

$$\int_A f(x, y)dxdy = \int_{\mathbb{R}^r} g(x)dx = \int_{\mathbb{R}^r} \left[\int_{A_x} f(x, y)dy \right] dx$$