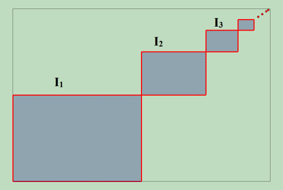
Real Analysis MAA 6616
Lecture 18
Tonelli's Theorem and Applications

Fubini's Theorem asserts that the integral of an integrable function in $\mathbb{R}^{r+s}$ is the same as the iterated integrals and interchange of order of integration is allowed. Tonelli's Theorem gives a converse for nonnegative functions (a nonnegative function with finite iterated integrals is integrable). It should be noted that this result does not generalize to functions that change sign as illustrated by the following example.

Let $I = [0,\ 1] \times [0,\ 1]$ be the unit square in $\mathbb{R}^2$. Consider the sequence of squares in $\{I_k\}_k$ in I given by:

$$I_1 = \left[0,\, \frac{1}{2}\right]^2, \quad I_k = \left[1 - \frac{1}{2^{k-1}},\ 1 - \frac{1}{2^k}\right]^2 \quad \text{for } k = 2, 3, \cdots$$

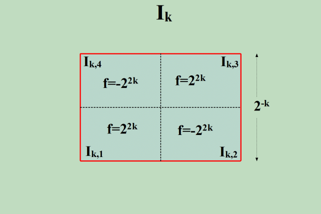


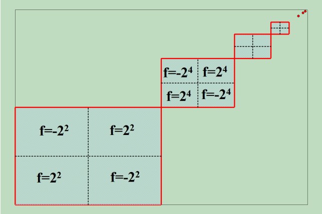
So that I_k has side length $\dfrac{1}{2^k}$ and area $\dfrac{1}{2^{2k}}$. Divide each square I_k into four equal squares $I_{k,1},\ I_{k,2},\ I_{k,3}$ and $I_{k,4}$ as in the figure bellow so that $I_{k,j}$ has side length $\dfrac{1}{2^{(k+1)}}$

Define a function $f : I \longrightarrow \mathbb{R}$ as follows:

$$f = 0 \text{ on } I \backslash \left(\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{4} I_{k,j}^{o} \right) \ , \ f = 2^{2k} \text{ in } I_{k,1}^{o} \cup I_{k,3}^{o} \ , \ \text{and } f = -2^{2k} \text{ in } I_{k,2}^{o} \cup I_{k,4}^{o}$$

where E^{o} denotes the interior of a set E. Hence $f = \sum_{k=1}^{\infty} \sum_{j=1}^{4} (-1)^{j+1} 2^{k} \chi_{I_{k,j}^{o}}$





The function f is measurable on I. For each $x \in [0,\ 1], f(x, \cdot) \in \mathcal{L}([0,\ 1])$ and for each $y \in [0,\ 1], f(\cdot, y) \in \mathcal{L}([0,\ 1])$. Furthermore,

$$\int_0^1 \left[\int_0^1 f(x,y)dy \right] dx = \int_0^1 \left[\int_0^1 f(x,y)dx \right] dy = 0.$$

However, $f \notin \mathcal{L}([0,\ 1] \times [0,\ 1])$ since

$$|f| = \sum_{k=1}^{\infty} \sum_{j=1}^{4} 2^{2k} \chi_{I_{k,j}} \quad \text{and} \quad \int_I |f| dxdy = \sum_{k=1}^{\infty} 1 = \infty \, .$$

Theorem (1. Tonelli's Theorem)

Let $E \subset \mathbb{R}^r$ and $F \subset \mathbb{R}^s$ be measurable sets and $f : E \times F \longrightarrow, \overline{\mathbb{R}}$ be a **nonnegative** *measurable function. Then, we have the followings:*

- *For a.e. $x \in E$, $f(x, \cdot)$ is measurable in F.*
- *For a.e. $y \in F$, $f(\cdot, y)$ is measurable in E.*
- *The function $g(x) = \displaystyle\int_F f(x, y)dy$ is measurable in E.*
- *The function $h(y) = \displaystyle\int_E f(x, y)dx$ is measurable in F.*
- *We have the following equality of the integrals(in $\overline{\mathbb{R}}$)*
$$\int_{E\times F} f(x, y)\, dxdy = \int_E \left[\int_F f(x, y)\, dy\right] dx = \int_F \left[\int_E f(x, y)\, dx\right] dy \qquad .$$

Proof.

This theorem is a consequence of Fubini's Theorem (FT) and the Monotone Convergence Theorem (MCT). We start by defining an increasing sequence $\{f_n\}_n$ of bounded, nonnegative, and integrable functions. For $n \in \mathbb{N}$ let C_n the cube in $\mathbb{R}^{r+s}$ given by $C_n = [-n,\ n]^{r+s}$. Define the function f_n by

$$f_n(x, y) = \begin{cases} \min(f(x, y), n) & \text{if } (x, y) \in C_n \\ 0 & \text{if } (x, y) \notin C_n \end{cases}$$

It is verified at once that $f_n \nearrow f$ on $E \times F$. Furthermore, it follows from FT that $f_n(x, \cdot) \in \mathcal{L}(\mathbb{R}^s)$ for a.e. $x \in \mathbb{R}^r$ and $g_n(x) = \displaystyle\int_{\mathbb{R}^s} f_n(x, y)dy \in \mathcal{L}(\mathbb{R}^r)$. Since $f(x, .)$ is nonnegative its integral, $g(x) = \displaystyle\int_F f(x, y)dy$ exists(but could be ∞).

$\square$

Proof.

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Now $f_n(x,\cdot) \nearrow f(x,\cdot)$, then the MCT Theorem implies that $g_n(x) \nearrow g(x) = \int_F f(x,y)dy$. for a.e. $x \in E$ and g is measurable in E. Applications of the MCT and FT give

$$\begin{aligned}
\int_E \left[\int_F f(x,y)\right] dx &= \int_E g(x)\, dx = \lim_{n\to\infty} \int_E g_n(x)\, dx && \text{(MCT)} \\
&= \lim_{n\to\infty} \int_E \left[\int_F f_n(x,y)\, dy\right] dx = \lim_{n\to\infty} \int_{E\times F} f_n(x,y)\, dxdy && \text{(FT)} \\
&= \int_{E\times F} f(x,y) dxdy && \text{(MCT)}
\end{aligned}$$

To complete the proof, interchange the roles of x and y. $\square$

Remark (1)

Tonelli's Theorem tells us that for nonnegative functions in $E \times F \subset \mathbb{R}^{r+s}$, the finiteness of any one of the three integrals

$$\int_{E\times F} f(x,y)dxdy\,; \quad \int_E \left[\int_F f(x,y)dy\right] dx\,; \quad \text{or} \quad \int_F \left[\int_E f(x,y)dx\right] dy\,,$$

implies the finiteness of the other two and the three integrals are equal.

Let $f, g \in \mathcal{L}(\mathbb{R}^n)$, then we define their convolution $f * g$ as the function in $\mathbb{R}^n$ given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt\,.$$

Provided that the integral exists. This operation of convolution is an important tool used in Analysis. We will show that the convolution of two integrable functions exists and that the convolution is commutative: $f * g = g * f$. Before going further, we need the following theorem.

Theorem (2)

Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map. That is there exists an $n \times n$ constant matrix A such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$. Then L maps a measurable set onto a measurable set. Moreover, if $E \subset \mathbb{R}^n$ is measurable, then $m(L(E)) = |\det(A)|\, m(E)$, where $\det(A)$ is the determinant of the matrix A.

Lemma (1)

Let f be a measurable function in $\mathbb{R}^n$, then the function $F(x,t) = f(x-t)$ is measurable in $\mathbb{R}^{2n}$.

Proof.

First consider the function $\tilde{f} : \mathbb{R}^{2n} \longrightarrow \overline{\mathbb{R}}$ given by $\tilde{f}(x,t) = f(x)$. Then $\tilde{f}$ is measurable. Indeed for $c \in \mathbb{R}$ we have $\{\tilde{f} > c\} = \{x \in \mathbb{R}^n : f(x) > c\} \times \mathbb{R}^n$ is a measurable in $\mathbb{R}^{2n}$ as a product of two measurable sets.

Let $L : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ be the nonsingular linear transformation given by $L(x,t) = (x-t, x+t)$. Then $F(x,t) = f(x-t) = \tilde{f} \circ L(x,t)$. The conclusion follows from Theorem 2. $\square$

Theorem (3)

Let $f,\ g\ \in \mathcal{L}(\mathbb{R}^n)$, then

1. $(f*g)(x)$ *exists for a.e.* $x \in \mathbb{R}^n$ *and* $f*g \in \mathcal{L}(\mathbb{R}^n)$*;*
2. $f*g = g*f$*;*
3. $\displaystyle\int_{\mathbb{R}^n} |f*g|\ dx \ \le\ \left(\int_{\mathbb{R}^n} |f|\ dx\right)\left(\int_{\mathbb{R}^n} |g|\ dx\right)$*;*
4. *if f and g are nonnegative, then* $\displaystyle\int_{\mathbb{R}^n} (f*g)(x)\, dx \ =\ \left(\int_{\mathbb{R}^n} f(x)\, dx\right)\left(\int_{\mathbb{R}^n} g(x)\, dx\right)$

Proof.

We start by using Tonelli's Theorem to $|f(t)g(x-t)|$ in $\mathbb{R}^{2n}$ to get

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(t)|\,|g(x-t)|\ dxdt \quad &= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |f(t)|\,|g(x-t)|\ dx\right]\ dt = \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |g(x-t)|\ dx\right] |f(t)|\ dt \\ &= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} |g(y)|\ dy\right] |f(t)|\ dt = \left(\int_{\mathbb{R}^n} |g(y)|\, dy\right)\left(\int_{\mathbb{R}^n} |f(t)|\ dt\right)\end{aligned}$$

Now that we verified that $f(t)g(x-t)\ \in\ \mathcal{L}(\mathbb{R}^{2n})$ we can apply Fubini's Theorem to obtain that $f(t)g(x-t)$ as a function of t is in $\mathcal{L}(\mathbb{R}^n)$ for a.e. $x \in \mathbb{R}^n$ and $(f*g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)\ dt$ exist for a.e. $x \in \mathbb{R}^n$.

A substitution in the integral defining the convolution shows that $*$ is a commutative operation.

The third claim of the theorem follows from $\left|\int_E F dx\right| \le \int_E |F|\ dx$ and the fourth from the above calculation where no absolute value is needed. $\square$