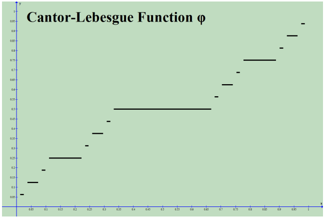
Real Analysis MAA 6616
Lecture 19
Continuity of Monotone Functions and Vitali Covering Lemma

We examine the relationship between integration and differentiation. The Fundamental Theorem of Calculus (FTC) states that if $f : [a,\ b] \longrightarrow \mathbb{R}$ is differentiable and if its derivative f' is continuous on $[a,\ b]$, then $\int_a^b f'(x)dx = f(b) - f(a)$.

A natural question is to understand if this result can be extended if we replace the differentiability of f on $[a,\ b]$ by differentiability of f a.e. on $[a,\ b]$ and continuity of f' by Lebesgue integrability of f'. In this general setting, the answer is NO and an example is given by the Cantor-Lebesgue function ϕ on $[0,\ 1]$. This function is increasing, ϕ' exists and is 0 a.e. on $[0,\ 1]$ so that

$$\left(\int_0^1 \phi'(x)dx = 0\right) \neq (\phi(1) - \phi(0) = 1)$$



We will prove that FTC holds for a large class of functions. First we need to understand the differentiability of monotone functions on an interval.

Theorem (1)

Let $f : (a,\ b) \longrightarrow \mathbb{R}$ be a monotone function. Then f is continuous on $(a,\ b)$ except possibly at a countable set of points.

Proof.

Without loss of generality, we can assume that $f \nearrow$.
Case : $(a,\ b)$ *bounded and* $f : [a,\ b] \longrightarrow \mathbb{R}$. For $x_0 \in (a,\ b)$ define

$$f(x_0^-) = \lim_{x \longrightarrow x_0^-} f(x) = \sup\{f(x) : a < x < x_0\} \text{ and } f(x_0^+) = \lim_{x \longrightarrow x_0^+} f(x) = \inf\{f(x) : x_0 < x < b\}.$$

Note that $f(x_0^{\pm})$ are finite and $f(a) \leq f(x_0^-) \leq f(x_0) \leq f(x_0^+) \leq f(b)$. Hence f is discontinuous at x_0 if and only if $f(x_0^+) - f(x_0^-) > 0$. In the case f is discontinuous at x_0, define the jump interval as $J(x_0) = \left(f(x_0^-),\ f(x_0^+)\right)$. Since $f \nearrow$, then $J(x_0) \subset [f(a),\ f(b)]$ and moreover if $x_1 < x_2$ are discontinuities of f, then $J(x_1) \cap J(x_2) = \emptyset$. For $n \in \mathbb{N}$, let E_n be the set of discontinuities x of f with $\ell(J(x)) > 1/n$. Then $E_n \subset [a,\ b]$ is covered by a collection of disjoint intervals each of which has length $> 1/n$. Therefore E_n must be a finite set. Let D be the set of all discontinuities of f. Then $D = \bigcup_{n=1}^{\infty} E_n$ is countable (countable union of finite sets).

In the case $(a,\ b)$ unbounded or $f(a^+)$, or $f(b^-)$ not finite, we can write $(a,\ b) = \bigcup_{m=1}^{\infty}\left[a + \frac{1}{m},\ b - \frac{1}{m}\right]$. If D_m is the set of discontinuities in $\left[a + \frac{1}{m},\ b - \frac{1}{m}\right]$, then D_m is countable and the set D of discontinuities in $(a,\ b)$ is $D = \bigcup_m D_m$ also a countable set. □

Proposition (1)

Let $C \subset (a,\ b)$ be a countable set. Then there exist an increasing function $f : (a,\ b) \longrightarrow \mathbb{R}$ such that C is the set of discontinuities of f.

The proof is left as an exercise

Lemma (1)

Let $E \subset \mathbb{R}^n$ with $m^*(E) < \infty$. Then for any constant $0 < \alpha < 5^{-n}$ and for any collection of cubes $\mathcal{C}$ in $\mathbb{R}^n$ covering E, there exists a finite number of disjoint cubes $C_1, \cdots, C_N \in \mathcal{C}$ such that

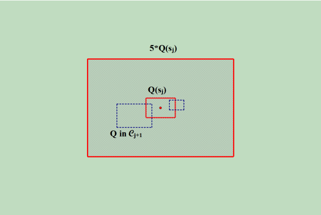
$$\sum_{j=1}^{N} \mathrm{vol}(C_j) \geq \alpha m^*(E)$$

Proof.

A cube $Q \in \mathcal{C}$ with side length s will be referenced as $Q(s)$ and we will use the notation $5 * Q$ to denote the cube with same center as Q and with side length $5s$ so that $\mathrm{vol}(5 * Q) = 5^n \mathrm{vol}(Q)$.
Set $\mathcal{C}_1 = \mathcal{C}$ and let $s_1^* = \sup\{s : Q(s) \in \mathcal{C}_1\}$. If $s_1^* = \infty$, then there exists a sequence $s_j \in \{s : Q(s) \in \mathcal{C}_1\}$ such that $\lim_{j\to\infty} s_j = \infty$. Since $m^*(E) < \infty$, then for every given $\alpha > 0$, we can find a cube $Q(s_j) \in \mathcal{C}_1$ such that $\mathrm{vol}(Q(s_j)) = s_j^n > \alpha m^*(E)$.
If $s_1^* < \infty$, then we can find $Q(s_1) \in \mathcal{C}_1$ with $s_1 > s_1^*/2$. Let

$$\mathcal{C}_2 = \{Q(s) \in \mathcal{C}_1 : Q(s) \cap Q(s_1) = \emptyset\} \quad \text{and} \quad \mathcal{C}_2' = \{Q(s) \in \mathcal{C}_1 : Q(s) \cap Q(s_1) \neq \emptyset\} = \mathcal{C}_1 \backslash \mathcal{C}_2.$$

Note that since $2s_1 > s_1^*$, then every cube $Q \in \mathcal{C}_2'$ is contained in $5 * Q(s_1)$ (verification left as an exercise).



Proof.

CONTINUED: Let $s_2^* = \sup\{s : Q(s) \in \mathcal{C}_2\}$ and let $Q(s_2) \in \mathcal{C}_2$ with $s_2 > s_2^*/2$. Let

$$\mathcal{C}_3 = \{Q(s) \in \mathcal{C}_2 : Q(s) \cap Q(s_2) = \emptyset\} \text{ and } \mathcal{C}_3' = \{Q(s) \in \mathcal{C}_2 : Q(s) \cap Q(s_2) \neq \emptyset\} = \mathcal{C}_2 \backslash \mathcal{C}_3.$$

Again, if $Q \in \mathcal{C}_3'$, then $Q \subset 5 * Q(s_2)$.

By induction, we construct families of cubes $\mathcal{C}_j$ and $\mathcal{C}_j'$ and a sequences of cubes $Q(s_j)$ such that:

- $Q(s_j) \in \mathcal{C}_j$ with $s_j > s_j^*/2$ with $s_j^* = \sup\{s : Q(s) \in \mathcal{C}_j\}$;
- $\mathcal{C}_{j+1} = \{Q(s) \in \mathcal{C}_j : Q(s) \cap Q(s_j) = \emptyset\}$ and $\mathcal{C}_{j+1}' = \mathcal{C}_j \backslash \mathcal{C}_{j+1}$

Note that the family of cubes $\{Q(s_j)\}_j$ is disjoint and every cube in $\mathcal{C}_{j+1}'$ is contained in $5 * Q(s_j)$.

We have a decreasing sequence $\{s_j^*\}_j$. If there exists N such that $\mathcal{C}_{N+1} = \emptyset$ (which means $s_{N+1}^* = 0$), the process ends and we have disjoint cubes $Q(s_1), \cdots, Q(s_N)$. Moreover, by construction we have

$$\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2 \cup \mathcal{C}_2' = \mathcal{C}_3 \cup \mathcal{C}_3' \cup \mathcal{C}_2' = \cdots = \mathcal{C}_{N+1} \cup \mathcal{C}_{N+1}' \cup \cdots \cup \mathcal{C}_2' = \mathcal{C}_{N+1}' \cup \cdots \cup \mathcal{C}_2'$$

Since $E \subset \bigcup_{Q \in \mathcal{C}_1} Q$, then E is contained in the union of all the cubes in $\mathcal{C}_{N+1}' \cup \cdots \cup \mathcal{C}_2'$. Since each cube in $\mathcal{C}_{j+1}'$ is contained in $5 * Q(s_j)$, then $E \subset \bigcup_{j=1}^{N} (5 * Q(s_j))$. Therefore

$$m^*(E) \leq \sum_{j=1}^{N} \text{vol}(5 * Q(s_j)) = 5^n \sum_{j=1}^{N} \text{vol}(Q(s_j))$$

and the lemma is proved in this case.

If $s_j^* > 0$ for all j, then we consider two situations: $s_j^* \searrow \delta$ with $\delta > 0$ and $s_j^* \searrow 0$. In the first situation, we have $s_j > \delta/2$ for all $j \in \mathbb{N}$. We have $\text{vol}(Q(s_j)) > (\delta/2)^n$ and for any given $\alpha > 0$ we can find N such that then $\sum_{j=1}^{N} \text{vol}(Q(s_j)) > \alpha m^*(E)$. Finally, if $s_j^* \searrow 0$, then every $Q \in \mathcal{C} = \mathcal{C}_1$ would be contained in $\bigcup_{j=1}^{\infty} (5 * Q(s_j))$.

Otherwise there would be $Q(s) \in \mathcal{C}$ $(s > 0)$ such that $Q(s) \cap Q(s_j) = \emptyset$ for all j, and this would mean $s \leq s_j^*$ for all j and so $s = 0$. Hence, $E \subset \bigcup_{j=1}^{\infty} (5 * Q(s_j))$ and so $m^*(E) \leq 5^n \sum_{j=1}^{\infty} \text{vol}(Q(s_j))$. Consequently, if $a < 5^{-n}$, then we can find N such that $\sum_{j=1}^{N} \text{vol}(Q(s_j)) > \alpha m^*(E)$

A collection of cubes $\mathcal{C}$ is said to cover a set E in the sense of Vitali if for for every $\delta > 0$ and for every $x \in E$ there exist a cube $Q \in \mathcal{C}$ with side length $< \delta$ such that $x \in Q$.

Theorem (2)

Suppose that a family of cubes $\mathcal{C}$ covers a set E in the Vitali sense. Suppose that $0 < m^*(E) < \infty$. Then for every $\epsilon > 0$, there exists a countable collection of disjoint cubes $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{C}$ such that

$$m\left(E \backslash \bigcup_{j=1}^{\infty} Q_j\right) = 0 \text{ and } m\left(\bigcup_{j=1}^{\infty} Q_j\right) \leq (1+\epsilon) m^*(E)$$

Proof.

Let $\alpha > 0$ such that $\alpha < 5^{-n}$ and let $\epsilon < \alpha$. Set $\gamma = 1 + \epsilon - \alpha$ so that $0 < \gamma < 1$. Let V be an open set such that $E \subset V$ and $m(V) < (1+\epsilon) m^*(E)$. Consider the family $\mathcal{C}_V \subset \mathcal{C}$ that consists of all cubes in $\mathcal{C}$ that are contained in V: $\mathcal{C}_V = \{Q \in \mathcal{C} : Q \subset V\}$. Then $\mathcal{C}_V$ is again a cover of E in the sense of Vitali.

It follows from the Simple Vitali Lemma that there exists N_1 disjoint cubes $Q_1, \cdots, Q_{N_1}$ in $\mathcal{C}_V$ such that $\sum_{j=1}^{N_1} \text{vol}(Q_j) > \alpha m^*(E)$. Let $Q_1^{N_1} = Q_1 \cup \cdots \cup Q_{N_1}$. We have then

$$m^*\left(E \backslash Q_1^{N_1}\right) \leq m\left(V \backslash Q_1^{N_1}\right) = m(V) - m(Q_1^{N_1}) \leq (1+\epsilon) m^*(E) - \alpha m^*(E) = \gamma m^*(E).$$

Let $E_1 = E \backslash Q_1^{N_1}$ and $\mathcal{C}_{1,V} = \mathcal{C}_V \backslash \{Q_1, \cdots, Q_{N_1}\}$. We have $m^*(E_1) \leq \gamma m^*(E)$ and $\mathcal{C}_{1,V}$ covers E_1 in the Vitali sense. We repeat this construction when E is replaced by E_1 and $\mathcal{C}_V$ replaced by $\mathcal{C}_{1,V}$ to produce disjoint cubes $Q_{N_1+1}, \cdots, Q_{N_2}$ in $\mathcal{C}_{1,V}$ such that $\sum_{j=N_1}^{N_2} \text{vol}(Q_j) > \alpha m^*(E_1)$. Let $Q_1^{N_2} = Q_1 \cup \cdots \cup Q_{N_2}$. We have then

$$m^*\left(E \backslash Q_1^{N_2}\right) = m^*\left(E_1 \backslash \bigcup_{j=N_1+1}^{N_2} Q_j\right) \leq \gamma m^*(E_1) \leq \gamma^2 m^*(E).$$

Proof.

CONTINUED: By repeating this process m times, we obtain disjoint cubes $Q_1, \cdots, Q_{N_m}$ such that

$$m^*\left(E \setminus \bigcup_{j=1}^{N_m} Q_j\right) \leq \gamma^m m^*(E) .$$

Since $0 < \gamma < 1$, then $\lim_{m\to\infty} \gamma^m = 0$, then the countable collection of disjoint cubes $\{Q_j\}_j$ satisfies $m^*\left(E \setminus \bigcup_{j=1}^{\infty} Q_j\right) = 0$. Furthermore, since $\bigcup_{j=1}^{\infty} Q_j \subset V$, then $m\left(\bigcup_{j=1}^{\infty} Q_j\right) \leq m(V) \leq (1+\epsilon)m^*(E)$. □

The following corollary is a consequence of the proof of the Vitali Covering Lemma.

Corollary (1)

Suppose that a family of cubes $\mathcal{C}$ covers a set E in the Vitali sense. Suppose that $0 < m^*(E) < \infty$. Then for every $\epsilon > 0$, there exists a finite collection of disjoint cubes $Q_1, \cdots, Q_N \in \mathcal{C}$ such that

$$m^*\left(E \setminus \bigcup_{j=1}^{N} Q_j\right) < \epsilon \quad \text{and} \quad m\left(\bigcup_{j=1}^{N} Q_j\right) \leq (1+\epsilon)m^*(E)$$