# Real Analysis MAA 6616
# Lecture 2
# Open sets; Closed sets; Borel sets

A set $U \subset \mathbb{R}$ is open if for every $x \in U$ there exists $\epsilon > 0$ such that $(x-\epsilon,\ x+\epsilon) \subset U$.
**Example.** For $a,\ b \in \mathbb{R}$ with $a < b$, the interval $(a,\ b)$ is open. Indeed for $x \in (a,\ b)$ let $\epsilon = \min(x-a, b-x)$, then $(x-\epsilon, x+\epsilon) \subset (a,\ b)$.
The interval $[a,\ b)$ is not open since for $x = a \in [a,\ b)$ there is no $\epsilon > 0$ such that $(a-\epsilon,\ a+\epsilon) \subset [a,\ b)$.
$\mathbb{R}$ and $\emptyset$ are open.

## Proposition

1. *The union of any collection of open sets is open.*
2. *The intersection of a finite collection of open sets is open.*

## Proof.

- Let $\Lambda$ be set and for every $\lambda \in \Lambda$, let $U_\lambda \subset \mathbb{R}$ be open. Set $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. We need to prove that $U$ is open. Let $x \in U$. There exists $\lambda_0 \in \Lambda$ such that $x \in U_{\lambda_0}$. Since $U_{\lambda_0}$ is open, then there exists $\epsilon_0 > 0$ such that $(x-\epsilon_0,\ x+\epsilon_0) \subset U_{\lambda_0}$. Then $(x-\epsilon_0,\ x+\epsilon_0) \subset U$

$\square$

- Let $U_1, \cdots, U_N$ be open sets in $\mathbb{R}$ and $V = \bigcap_{j=1}^{N} U_j$. Let $x \in V$, then $x \in U_j$ for $j = 1, \cdots, N$. Hence for every $j \in \{1, \cdots, N\}$ there exists $\epsilon_j > 0$ such that $(x - \epsilon_j,\ x + \epsilon_j) \subset U_j$. Let $\epsilon_0 = \min_{1 \leq j \leq N} (\epsilon_j)$.

$$(x - \epsilon_0,\ x + \epsilon_0) \subset (x - \epsilon_j,\ x + \epsilon_j) \subset U_j \qquad \forall j \in \{1, \cdots, N\}$$

  Therefore $(x - \epsilon_0,\ x + \epsilon_0) \subset V$ and $V$ is open.

## Remark

Intersection of infinitely many open sets may not be open. For example for $n \in \mathbb{N}$, consider the open interval $I_n = (-\frac{1}{n}, \frac{1}{n})$. The intersection $\bigcap_{n=1}^{\infty} I_n = \{0\}$ is not open.

## Proposition

*A nonempty open set in $\mathbb{R}$ is the disjoint union of a countable collection of open intervals. More precisely, if $U$ is an open subset of $\mathbb{R}$, then there exists a countable set $\Lambda$, such that for each $\lambda \in \Lambda$ there is an open interval $I_\lambda \subset \mathbb{R}$ satisfying $I_\lambda \cap I_\mu = \emptyset$ if $\lambda \neq \mu$ and $U = \bigcup_{\lambda \in \Lambda} I_\lambda$.*

# Proof.

Let $U \subset \mathbb{R}$ be open. Define the relation $\sim$ in $U$ by: $x \sim y$ if and only if there exists an open interval $I \subset U$ such that $x, y \in I$. The relation $\sim$ is an equivalence relation (verification left as an exercise). For each $x \in U$, let $I(x)$ the equivalence class of $x$. Let $a(x) = \inf(I(x))$ and $b(x) = \sup(I(x))$. (note $a(x)$ could be $-\infty$ and $b(x)$ could be $\infty$).

We claim that $I(x) = (a(x),\ b(x))$ ($I(x)$ is the largest open interval in $U$ containing $x$). Indeed, let $y \in (a(x),\ b(x))$. by definition of l.u.b and g.l.b., there exist real numbers $u, v \in I(x)$ such that $u < y < v$. By definition of $\sim$ and of $I(x)$, there exist open intervals $I \subset U$ containing $u$ and $v$ and therefore containing also $y$. this means $y \sim x$ and $y \in I(x)$. Therefore $(a(x),\ b(x)) \subset I(x)$. Now if there exists $z \in I(x) \backslash (a(x),\ b(x))$, then either $z \leq a(x)$ or $z \geq b(x)$. But $z$ cannot be $< a(x)$ nor $> b(x)$ since $a(x)$ and $b(x)$ are the g.l.b. and l.u.b. of $I(x)$. Also $z$ cannot be equal to either $a(x)$ nor $b(x)$ since otherwise it would mean that $a(x) \sim x$ or $b(x) \sim x$ and leads to a contradiction. Hence $I(x) = (a(x),\ b(x))$.

We have then proved that $U = \bigcup_{x \in U} I(x)$ is a disjoint union of intervals (if $I(x) \neq I(y)$, then $I(x) \cap I(y) = \emptyset$ since these are equivalent classes). We need only to verify that the collection of intervals $I(x)$ is countable. In each equivalence class $I(x)$, we can select a rational number $r$ ($\mathbb{Q}$ is dense in $\mathbb{R}$) and the collection of equivalence classes is countable. $\square$

Let $E \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a closure point of $E$ if for every $\epsilon > 0$ $(x - \epsilon,\ x + \epsilon) \cap E \neq \emptyset$.
For example 0 and 1 are closure points for the interval $(0,\ 1]$.
The set of all closure points of $E$ is called the closure of $E$ and denoted $\overline{E}$. Note that $E \subset \overline{E}$.
A set $E$ is said to be closed if $E = \overline{E}$.

## Proposition

1. *Let $E \subset \mathbb{R}$. Then $\overline{E}$ is closed: $\overline{\overline{E}} = \overline{E}$.*
2. *$\overline{E}$ is the smallest closed set containing $E$: If $F \subset \mathbb{R}$ is closed and $E \subset F$, then $\overline{E} \subset F$.*

## Proof.

1. Let $z \in \overline{\overline{E}}$. Then for any $\epsilon > 0$, there exists $y \in (z - \epsilon,\ z + \epsilon) \cap \overline{E}$. Let $\epsilon' = \min(z + \epsilon - y,\ y - z + \epsilon)$. Since $y \in \overline{E}$, then there exists $x \in E \cap (y - \epsilon',\ y + \epsilon')$. It follows from the choice of $\epsilon'$ that $x \in E \cap (z - \epsilon,\ z + \epsilon)$. Since $\epsilon > 0$ is arbitrary, then $z \in \overline{E}$. Therefore $\overline{\overline{E}} \subset \overline{E}$ and so $\overline{E} = \overline{\overline{E}}$.
2. Let $F \subset \mathbb{R}$ be a closed set such that $E \subset F$. Let $y \in \overline{E}$. If $y \in E$, then $y \in F$. If $y \in \overline{E} \backslash E$, then for $\epsilon > 0$, arbitrary, there exists $x \in E \cap (y - \epsilon,\ y + \epsilon)$ and so $x \in F \cap (y - \epsilon,\ y + \epsilon)$. This means $y \in \overline{F} = F$ ($F$ closed) and $\overline{E} \subset F$.

$\square$

## Proposition

*A set $E \subset \mathbb{R}$ is closed if and only if its complement $\mathbb{R} \backslash E$ is open*

The proof is left as an exercise
It follows from the proposition that since $\mathbb{R}$ and $\emptyset$ are open, their complements $\mathbb{R} \backslash \mathbb{R} = \emptyset$ and $\mathbb{R} \backslash \emptyset = \mathbb{R}$ are closed.

## Proposition

1. *The union of a finite collection of closed sets is closed.*
2. *The intersection of any collection of closed sets is closed.*

## Proof.

1. Left as an exercise.
2. Let $\Lambda$ be a set and for each $\lambda \in \Lambda$, let $F_\lambda$ be a closed subset of $\mathbb{R}$. Let $F = \bigcap_{\lambda \in \Lambda} F_\lambda$. Since $F_\lambda$ is closed, then $U_\lambda = \mathbb{R} \backslash F_\lambda$ is open and $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ is open. We have

$$F = \bigcap_{\lambda \in \Lambda} F_\lambda = \bigcap_{\lambda \in \Lambda} (\mathbb{R} \backslash U_\lambda) = \mathbb{R} \backslash \left( \bigcup_{\lambda \in \Lambda} U_\lambda \right) = \mathbb{R} \backslash U$$

is closed since $U$ is open.

□

A cover of a set $E$ is a collection of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ such that $E \subset \bigcup_{\lambda\in\Lambda} E_\lambda$. A subcover is a subcollection $\{E_\mu\}_{\mu\in\Lambda'}$ such that $\Lambda' \subset \Lambda$ and $\{E_\mu\}_{\mu\in\Lambda'}$ is a cover of $E$. If each set $E_\lambda$ is open, then $\{E_\lambda\}_{\lambda\in\Lambda}$ is said to be an open cover of $E$. If $\Lambda$ is a finite set, the cover $\{E_\lambda\}_{\lambda\in\Lambda}$ is said to be a finite cover of $E$.

## Theorem

*Let $F \subset \mathbb{R}$ be a closed and bounded set. Then every open cover of $F$ has a finite subcover*

## Proof.

- **Case: $F = [a,\ b]$ a closed bounded interval.** Let $\mathcal{F} = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $F$. Consider the set $E \subset F$ defined as the set of points $x \in [a,\ b]$ such that the interval $[a,\ x]$ can be covered by a finite number of open set in $\mathcal{F}$. Since $\mathcal{F}$ is an open cover of $[a,\ b]$, then there exists $U \in \mathcal{F}$ containing $a$. Hence $E \neq \emptyset$. The set $E$ is bounded above by $b$. Let $s = \sup(E) \leq b$. We claim that $s = b$. Indeed, if $s < b$, then there is an open set $V \subset \mathcal{F}$ containing $s$ and $\epsilon > 0$ such that $(s-\epsilon,\ s+\epsilon) \subset V$. Since $s-\epsilon$ is not an upper bound of $E$, then there exists $x \in E$ and $s - \epsilon < x \leq s$. Since, $x \in E$, then $[a,\ x]$ can be covered by finitely many open sets $U_1, \cdots, U_n \in \mathcal{F}$. and so

  $$[a, s+\epsilon) \subset V \cup U_1 \cup \cdots \cup U_n .$$

  In this case we have a point $z$ with $s < z \leq b$ such that $[a,\ z]$ is covered by a finite number of open sets in $\mathcal{F}$ and this contradicts the definition of $s$. Hence $E = [a,\ b] = F$ and the Theorem is proved in this case.

# Proof.

- **General case: $F$ closed and bounded.** Let $\mathcal{F} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $F$. There exists a closed and bounded interval $[a,\ b]$ such that $F \subset [a,\ b]$. Let $V = \mathbb{R} \backslash F$. $V$ is open since $F$ is closed. Consider the collection of open sets $\mathcal{F}^* = \mathcal{F} \cup \{V\}$. Since $\mathcal{F}$ is a cover of $F$ and $([a,\ b] \backslash F) \subset V$, then $\mathcal{F}^*$ is an open cover of the interval $[a,\ b]$.
  The previous case implies that there exist finitely many open sets $U_1, \cdots, U_n \in \mathcal{F}$ such that $\{V, U_1, \cdots, U_n\}$ is a finite open cover of the interval $[a,\ b]$. Since $F \subset [a,\ b]$ and $V \cap F = \emptyset$, then $F \subset \bigcup_{j=1}^{n} U_j$ and the Theorem is proved.

□

A countable collection of sets $\{E_n\}_{n=1}^{\infty}$ is said to be nested or descending if $E_{n+1} \subset E_n$ for every $n \in \mathbb{N}$. The collection is called ascending if $E_n \subset E_{n+1}$ for every $n \in \mathbb{N}$.

## Theorem

*Let $\{F_n\}_{n=1}^{\infty}$ be a countable collection of nested, closed and bounded subsets of $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

## Proof.

By contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. For each $n \in \mathbb{N}$, let $U_n = \mathbb{R} \backslash F_n$, then $U_n$ is open since $F_n$ is closed. Furthermore, it follows from $\{F_n\}_{n=1}^{\infty}$ a descending family that $\{U_n\}_{n=1}^{\infty}$ is an ascending collection ($U_n \subset U_{n+1}$). It follows from $\bigcap_{n=1}^{\infty} F_n = \emptyset$ that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x \notin F_n$ and so $x \in U_n$. Hence $\mathbb{R} = \bigcup_{n=1}^{\infty} U_n$.
As a consequence, $\{U_n\}_{n=1}^{\infty}$ is an open cover of the closed and bounded set $F_1$. Heine-Borel Theorem implies that it has a finite subcover: There is $p \in \mathbb{N}$ such that $F_1 \subset U_1 \cup \cdots \cup U_p = U_p$ (because $\{U_n\}_{n=1}^{\infty}$ is an ascending collection). This means $F_1 \subset \mathbb{R} \backslash F_p$ and therefore $F_p \not\subseteq F_1$ and this contradicts the nestedness of the collection $\{F_n\}_{n=1}^{\infty}$. □

## $\sigma$-algebra

Given a set $X$, a collection $\mathcal{A}$ of subsets of $X$ is called a $\sigma$-algebra of $X$, if

1. it contains the empty set: $\emptyset \in \mathcal{A}$;
2. it is closed under complement: if $E \in \mathcal{A}$, then $X \backslash E \in \mathcal{A}$; and
3. it is closed under countable union: if $E_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

- $\{\emptyset, X\}$ is a $\sigma$-algebra. It is contained in any $\sigma$-algebra $\mathcal{A}$ of $X$.
- The set of all subsets of $X$ (denoted $2^X$) is a $\sigma$-algebra of $X$. It contains all $\sigma$-algebras of $X$.
- a $\sigma$-algebra $\mathcal{A}$ of $X$ is closed under countable intersection. If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$.Indeed, $F_n = X \backslash E_n \in \mathcal{A}$ by condition (2) and condition (3) gives

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} (X \backslash E_n) = \bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$$

- Given a family $\mathcal{F}$ of subsets of $X$. The intersection of all $\sigma$-algebras $\mathcal{B}$ containing $\mathcal{F}$ ($\mathcal{F} \subset \mathcal{B}$) is a $\sigma$-algebra $\mathcal{A}$. It is the $\sigma$-algebra generated by the family $\mathcal{F}$.

Let $\{E_n\}_{n=1}^{\infty}$ be a countable collections of sets in a set $X$. Define the sets

$$\limsup\{E_n\}_{n=1}^{\infty} = \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} E_n\right) \quad \text{and} \quad \liminf\{E_n\}_{n=1}^{\infty} = \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} E_n\right)$$

- $x \in \limsup\{E_n\}_{n=1}^{\infty}$ means that for every $k \in \mathbb{N}$ there exists $n \geq k$ such that $x \in E_n$. Thus $x$ belongs to a infinitely many sets $E_n$.
- $x \in \liminf\{E_n\}_{n=1}^{\infty}$ means that there exists $k \in \mathbb{N}$ such that $x \in E_n$ for every $n \geq k$. Hence $x$ belongs to every $E_n$ except possibly for finitely many.
- $\liminf\{E_n\}_{n=1}^{\infty} \subset \limsup\{E_n\}_{n=1}^{\infty}$
- If $\mathcal{A}$ is a $\sigma$-algebra of $X$ and $\{E_n\}_{n=1}^{\infty} \subset \mathcal{A}$, then $\limsup\{E_n\}_{n=1}^{\infty} \in \mathcal{A}$ and $\liminf\{E_n\}_{n=1}^{\infty} \in \mathcal{A}$

The Borel $\sigma$-algebra of $\mathbb{R}$ is the $\sigma$-algebra $\mathcal{B}$ generated by the collection of all open subsets of $\mathbb{R}$.

- It follows from the definitions that every open set and every closed set in $\mathbb{R}$ is a Borel set. In particular a finite set or a countable set in $\mathbb{R}$ is Borel.
- A countable intersection of open sets in $\mathbb{R}$ is a Borel set (such a set is called a $G_\delta$-set); and a countable union of closed sets in$\mathbb{R}$ is a Borel set (such a set is called an $F_\sigma$-set).
- If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{B}$, then $\limsup\{E_n\}_{n=1}^{\infty} \in \mathcal{B}$ and $\liminf\{E_n\}_{n=1}^{\infty} \in \mathcal{B}$