Real Analysis MAA 6616
Lecture 20
Differentiability of Monotone Functions

We start with the following observation. Suppose that a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable. If on an interval $I = (m,\ n)$ we have $g'(x) > c$, then $g(n) - g(m) = \int_m^n g'(x)dx \geq c(m-n)$. This can be written as

$$\ell(I) \leq \frac{1}{c}\left(g(n) - g(m)\right)$$

The first result is a sort of generalization of this version of the Mean Value Theorem to increasing function. First we need to defined upper and lower derivatives.

Let $f : E \subset \mathbb{R} \longrightarrow \overline{\mathbb{R}}$. For a point x in the interior of E, the upper derivative $D^+f(x)$ of f at x and the lower derivative $D^-f(x)$ of f at x are defined by:

$$D^+f(x) = \lim_{h \to 0}\left[\sup_{0<|t|<h} \frac{f(x+t) - f(x)}{t}\right]$$

and

$$D^-f(x) = \lim_{h \to 0}\left[\inf_{0<|t|<h} \frac{f(x+t) - f(x)}{t}\right]$$

Note that for an interior point $D^{\pm}f(x)$ always exist in $\overline{\mathbb{R}}$ and $D^+f(x) \geq D^-f(x)$. The function is differentiable $D^+f(x) = D^-f(x)$ and are finite. In this case the common value is the derivative $f'(x)$.

Lemma (1)

Let f be an increasing function on an interval $[a,\ b]$. Then for every $\gamma > 0$, we have

$$m^*\left(\left\{x \in (a,\ b) : D^+f(x) \geq \gamma\right\}\right) \leq \frac{1}{\gamma}(f(b) - f(a))$$

and

$$m^*\left(\left\{x \in (a,\ b) : D^+f(x) = \infty\right\}\right) = 0$$

Proof.

For $\gamma > 0$, let $E_\gamma = \{x \in (a,\ b) : D^+ f(x) \geq \gamma\}$. Let β be any real number such that $0 < \beta < \gamma$ and consider the family of intervals $\mathcal{C} = \{[r,\ s] \subset (a,\ b) : f(s) - f(r) \geq \beta(s - r)\}$. Then $\mathcal{C}$ is a cover of E_γ in the sense of Vitali. Indeed, for $x \in E_\gamma$ and foer $\epsilon > 0$, it follows from $D^+ f(x) \geq \gamma$, that there exists $t \in \mathbb{R}$ with $0 < |t| < \epsilon$ such that $\dfrac{f(x+t) - f(x)}{t} \geq \gamma$. Therefore, $[x,\ x+t] \in \mathcal{C}$ if $t > 0$ and $[x+t,\ x] \in \mathcal{C}$ if $t < 0$. We have then (from the Vitali Covering Lemma) the existence a finite number of disjoint intervals $\{[r_k,\ s_k]\}_{k=1}^N \subset \mathcal{C}$ such that:

$$m^* \left(E_\gamma \setminus \bigcup_{k=1}^N [r_k,\ s_k] \right) < \epsilon\ .$$

Since

$$E_\gamma \ \subset \ \bigcup_{k=1}^N [r_k,\ s_k] \cup \left(E_\gamma \setminus \bigcup_{k=1}^N [r_k,\ s_k] \right)$$

we have

$$m^* \left(E_\gamma \right) \leq \sum_{k=1}^N (s_k - r_k) + \epsilon \leq \frac{1}{\beta} \sum_{k=1}^N (f(s_k) - f(r_k)) + \epsilon \leq \frac{1}{\beta} (f(b) - f(a)) + \epsilon.$$

The last inequality follows from f increasing. Since $\epsilon > 0$ and β are arbitrary ($0 < \beta < \gamma$), then $m^* \left(E_\gamma \right) \leq \dfrac{1}{\gamma} (f(b) - f(a))$.

For the second part of the lemma note that for any $n \in \mathbb{N}$, we have $\left\{ x \in (a,\ b) : D^+ f(x) = \infty \right\} \subset E_n$. Thus

$$m^* \left(\left\{ x \in (a,\ b) : D^+ f(x) = \infty \right\} \right) \leq m^* \left(E_n \right) \leq \frac{f(b) - f(a)}{n}\ .$$

This completes the proof. □

Theorem (1)

Let $f : (a,\ b) \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a monotone function. Then f is differentiable almost everywhere in $(a,\ b)$.

Proof.

Without loss of generality, we can assume that f is increasing. First consider $(a,\ b)$ bounded. Let E be the set of points where f is not differentiable: $E = \{x \in (a,\ b) : D^+ f(x) > D^- f(x)\}$.
For each pair of rational numbers α, β with $\alpha > \beta$, let

$$E_{\alpha,\beta} = \{x \in (a,\ b) : D^+ f(x) > \alpha > \beta > D^- f(x)\}.$$

Then $E_{\alpha,\beta} \subset E$ and $E = \bigcup_{\alpha,\beta \in \mathbb{Q}} E_{\alpha,\beta}$.

Now we prove that $m^*\left(E_{\alpha,\beta}\right) = 0$. For this, let $\epsilon > 0$. There exists an open set U such that $E_{\alpha,\beta} \subset U \subset (a,\ b)$ and $m(U) \leq m^*\left(E_{\alpha,\beta}\right) + \epsilon$. Let $\mathcal{C}_\beta$ be the collection of all closed intervals $[u,\ v] \subset U$ such that $f(v) - f(u) < \beta(v - u)$.
Note that $\mathcal{C}_\beta$ is a cover in the Vitali sense for the set $E_{\alpha,\beta}$. Indeed if $x \in E_{\alpha,\beta}$, then $D^- f(x) < \beta$ and it follows from the definition of the lower derivative that for every $h > 0$ $|f(x+t) - f(x)| < \beta|t|$ for some t with $0 < |t| < h$. Hence $[x,\ x+t] \in \mathcal{C}_\beta$ if $t > 0$ and $[x+t,\ x] \in \mathcal{C}_\beta$ if $t < 0$.
It follows from the Vitali Covering Lemma that there exists a finite collection of disjoint intervals $I_k = [u_k,\ v_k]$, $k = 1, \cdots, N$, contained in $\mathcal{C}_\beta$ such that $m^*\left(E_{\alpha,\beta} \backslash I^N\right) < \epsilon$, where $I^N = I_1 \cup \cdots \cup I_N$. Inequality $m(U) \leq m^*\left(E_{\alpha,\beta}\right) + \epsilon$ together with $I_1^N \subset U$ and $I_k \in \mathcal{C}_\beta$ for $k = 1, \cdots, N$ imply that

$$\sum_{k=1}^{N}(f(v_k) - f(u_k)) \leq \beta\left(\sum_{k=1}^{N}(v_k - u_k)\right) = \beta m(I_1^N) \leq \beta m(U) \leq \beta m^*\left(E_{\alpha,\beta}\right) + \beta\epsilon$$

Lemma 1 can be applied to the function f on each interval I_k to the set $E_{\alpha,\beta} \cap I_k$ and we get $m^*\left(E_{\alpha,\beta} \cap I_k\right) \leq \frac{1}{\alpha}[f(v_k) - f(u_k)]$. It follows that ☐

Proof.

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$$\begin{aligned} m^*\left(E_{\alpha,\beta}\right) &\leq m^*\left(E_{\alpha,\beta}\cap I^N\right)+m^*\left(E_{\alpha,\beta}\backslash I^N\right)\leq \sum_{k=1}^{N} m^*\left(E_{\alpha,\beta}\cap I_k\right)+m^*\left(E_{\alpha,\beta}\backslash I^N\right)\\ &\leq \frac{1}{\alpha}\sum_{k=1}^{N}[f(v_k)-f(u_k)]+\epsilon\leq \frac{1}{\alpha}\left(\beta m^*\left(E_{\alpha,\beta}\right)+\beta\epsilon\right)+\epsilon\\ &\leq \frac{\beta}{\alpha}m^*\left(E_{\alpha,\beta}\right)+\frac{1+\alpha}{\alpha}\epsilon \end{aligned}$$

From this inequality we deduce that for any given rational numbers $\alpha > \beta$ and for any $\epsilon > 0$, we have $m^*\left(E_{\alpha,\beta}\right) \leq \dfrac{1+\alpha}{\alpha-\beta}\epsilon$. This means $m(E_{\alpha,\beta}) = 0$. Consequently, the set of points E where f is not differentiable has measure 0 (since E is the (countable) union of the sets $E_{\alpha,\beta}$).

If the interval $I = (a,\ b)$ is unbounded, let $n \in \mathbb{N}$ and $J_n = (a,b) \cap (-n,\ n)$. Then $I = \bigcup_{n=1}^{\infty} J_n$. Let E, the set of points in I, where f is not differentiable. Then $E \cap J_n$ has measure 0 (previous case), therefore $m(E) = 0$. □

Lemma (2)

Let $E \subset \mathbb{R}$. Then E has measure 0 if and only if there exists a countable family of open intervals $\{I_j\}_j$ such that $\sum_{j=1}^{\infty} \ell(I_j) < \infty$ and every $x \in E$ is contained in infinitely many intervals I_j.

Proof.

"$\Longleftarrow$" Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} \ell(I_j) < \epsilon$. Let $\delta_N = \min\{\ell(I_j) : j = 1, \cdots, N\}$. For every $x \in E$, there exists $I(x) \in \{I_j\}_j$ with $\ell(I(x)) < \delta_N$. Therefore $E \subset \bigcup_{j=N+1}^{\infty} I_j$ and so $m^*(E) \leq \sum_{j=N+1}^{\infty} \ell(I_j) < \epsilon$. Hence $m(E) = 0$. "$\Longrightarrow$" If $m(E) = 0$, then for any $n \in \mathbb{N}$, there exists an open set $U_n \supset E$ such that $m(U_n) < 2^{-n}$. There exists a countable collection of disjoint open intervals $\{I_j^n\}_{j=1}^{\infty}$ such that $U_n = \bigcup_{j=1}^{\infty} I_j^n$. The countable collection of open intervals $\{I_j^n\}_{j,n\in\mathbb{N}}$ is such that every $x \in E$ is contained in infinitely many I_j^n's and $\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_j^n) \leq \sum_{n=1}^{\infty} 2^{-n} = 1$ □

Theorem (2)

Let $E \subset (a,\ b)$ be a set with measure 0. Then there exists an increasing function $f : (a,\ b) \longrightarrow \mathbb{R}$ such that $f'(x)$ does not exist for all $x \in E$.

Proof.

Since $m(E) = 0$, then we can find a countable collection of open intervals $I_k = (u_k,\ v_k)$, $k \in \mathbb{N}$, such that $\sum_{k=1}^{\infty} \ell(I_k) < \infty$ and every $x \in E$ is contained in infinitely many intervals I_k.
Define $f : (a,\ b) \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \ell\left(I_k \cap (-\infty,\ x)\right) .$$

The function f is well defined since $\sum_{k=1}^{\infty} \ell(I_k) < \infty$ and it is increasing. If $x_1 < x_2$, then $\ell\left(I_k \cap (-\infty,\ x_1)\right) \leq \ell\left(I_k \cap (-\infty,\ x)\right)$ and so $f(x_1) \leq f(x_2)$.

Note that $\ell(I_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Let $x \in E$. So x is in infinitely many I_k's. Then for any given $h > 0$ and for any given $N \in \mathbb{N}$, there exists t with $0 < t < h$ such that $[x,\ x+t]$ is contained in at least N intervals I_k. It follows that

$$f(x+t) - f(x) = \sum_{k=1}^{\infty} \ell\left(I_k \cap (-\infty,\ x+t)\right) - \sum_{k=1}^{\infty} \ell\left(I_k \cap (-\infty,\ x)\right) = \sum_{k=1}^{\infty} \ell\left(I_k \cap (x,\ x+t)\right) \geq Nt$$

This implies $D^{+} f(x) \geq N$. Since $N \in \mathbb{N}$ is arbitrary, then $D^{+} f(x) = \infty$ and f is not differentiable at $x \in E$. □

Theorem (3)

Let $f : [a,\ b] \longrightarrow \mathbb{R}$ be an increasing function. The derivative f' is nonnegative, measurable, and is in $\mathcal{L}((a,\ b))$. Furthermore

$$\int_a^b f'(x)dx \leq f(b) - f(a)$$

Proof.

We know from Theorem 1 that $f'(x)$ exist for a.e. $x \in (a, b)$. That $f'(x)$ is nonnegative follows from the increase of f which implies that the difference quotient $\dfrac{f(x+t)-f(x)}{t}$ is nonnegative for all $t \neq 0$.

Extend the function f to the interval $(a, b+1)$ by defining it on $[b, b+1)$ as the constant $f(b)$. Consider the sequence of functions f_n on $[a, b]$ given by

$$f_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} = n\left(f(x+\frac{1}{n})-f(x)\right)$$

Then $f_n \longrightarrow f'$ pointwise a.e. in $[a, b]$. Since f_n is nonnegative and measurable, then f' is measurable. By Fatou's Lemma we have

$$\int_a^b f'(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_a^b f_n(x)dx.$$

We have

$$\begin{aligned}\int_a^b f_n(x)dx &= n\int_a^b f(x+\frac{1}{n})dx - n\int_a^b f(x)dx = n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - n\int_a^b f(x)dx \\ &= n\int_b^{b+\frac{1}{n}} f(x)dx - n\int_a^{a+\frac{1}{n}} f(x)dx \\ &\leq n\int_b^{b+\frac{1}{n}} f(b)dx - n\int_a^{a+\frac{1}{n}} f(a)dx = f(b)-f(a)\end{aligned}$$

Therefore

$$\int_a^b f'(x)dx \leq \liminf_{n\to\infty} \int_a^b f_n(x)dx \leq f(b)-f(a).$$

□

Remark (1)

The inequality $\int_a^b f'(x)dx \leq f(b)-f(a)$ could be a strict inequality even when f is continuous. This is the case of the Cantor-Lebesgue function ϕ:

$$\int_0^1 \phi'(x)dx = 0 \ \text{ and } \ \phi(1)-\phi(0) = 1$$